

# The Superfluid Universe

G.E. Volovik

*Low Temperature Laboratory, Aalto University, Finland  
L.D. Landau Institute for Theoretical Physics, Moscow, Russia*

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## Abstract

We discuss phenomenology of quantum vacuum. Phenomenology of macroscopic systems has three sources: thermodynamics, topology and symmetry. Thermodynamics of the self-sustained vacuum allows us to treat the problems related to the vacuum energy: the cosmological constant problems. The natural value of the energy density of the equilibrium self-sustained vacuum is zero. Cosmology is discussed as the process of relaxation of vacuum towards the equilibrium state. The present value of the cosmological constant is very small compared to the Planck scale, because the present Universe is very old and thus is close to equilibrium.

Momentum space topology determines the universality classes of fermionic vacua. The Standard Model vacuum both in its massless and massive states is topological medium. The vacuum in its massless state shares the properties of superfluid  $^3\text{He-A}$ , which is topological superfluid. It belongs to the Fermi-point universality class, which has topologically protected fermionic quasiparticles. At low energy they behave as relativistic massless Weyl fermions. Gauge fields and gravity emerge together with Weyl fermions at low energy. This allows us to treat the hierarchy problem in Standard Model: the masses of elementary particles are very small compared to the Planck scale because the natural value of the quark and lepton masses is zero. The small nonzero masses appear in the infrared region, where the quantum vacuum acquires the properties of another topological superfluid,  $^3\text{He-B}$ , and 3+1 topological insulators. The other topological media in dimensions 2+1 and 3+1 are also discussed. In most cases, topology is supported by discrete symmetry of the underlying microscopic system, which indicates the important role of discrete symmetry in Standard Model.

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## I. INTRODUCTION. PHENOMENOLOGY OF QUANTUM VACUUM

### A. Vacuum as macroscopic many-body system

The aether of the 21-st century is the quantum vacuum. The quantum aether is a new form of matter. This substance has a very peculiar properties strikingly different from the other forms of matter (solids, liquids, gases, plasmas, Bose condensates, radiation, etc.) and from all the old aethers. The new aether has equation of state  $p = -\epsilon$ ; it is Lorentz invariant; and as follows from the recent cosmological observations its energy density is about  $10^{-29}\text{g/cm}^3$  (i.e. the quantum aether by 29 orders magnitude lighter than water) and it is actually anti-gravitating.

Quantum vacuum can be viewed as a macroscopic many-body system. Characteristic energy scale in our vacuum (analog of atomic scale in quantum liquids) is Planck energy  $E_P = (\hbar c^5/G)^{1/2} \sim 10^{19} \text{ GeV} \sim 10^{32}\text{K}$ . Our present Universe has extremely low energies and temperatures compared to the Planck scale: even the highest energy in the nowadays accelerators is extremely small compared to Planck energy:  $E_{\text{max}} \sim 10 \text{ TeV} \sim 10^{17}\text{K} \sim 10^{-15}E_P$ . The temperature of cosmic background radiation is much smaller,  $T_{\text{CMBR}} \sim 1 \text{ K} \sim 10^{-32}E_P$ .

Cosmology belongs to ultra-low frequency physics. Expansion of Universe is extremely slow: the Hubble parameter compared to the characteristic Planck frequency  $\omega_P = (c^5/G\hbar)^{1/2}$  is  $H \sim 10^{-60}\omega_P$ . This also means that at the moment our Universe is extremely close to equilibrium. This is natural for any many-body system: if there is no energy flux from environment the energy will be radiated away and the system will be approaching the equilibrium state with vanishing temperature and motion.

According to Landau, though the macroscopic many-body system can be very complicated, at low energy and temperatures its description is highly simplified. Its behavior can

be described in a fully phenomenological way, using the symmetry and thermodynamic consideration. Later it became clear that another factor also governs the low energy properties of a macroscopic system – topology. The quantum vacuum is probably a very complicated system. However, using these three sources – thermodynamics, symmetry and topology – we may try to construct the phenomenological theory of the quantum vacuum near its equilibrium state.

### B. 3 sources of phenomenology: thermodynamics, symmetry and topology

Following Landau, at low energy  $E \ll E_P$  the macroscopic quantum system – superfluid liquid or our Universe – contains two main components: vacuum (the ground state) and matter (fermionic and bosonic quasiparticles above the ground state). The physical laws which govern the matter component are more or less clear to us, because we are able to make experiments in the low-energy region and construct the theory. The quantum vacuum occupies the Planckian and trans-Planckian energy scales and it is governed by the microscopic (trans-Planckian) physics which is still unknown. However, using our experience with a similar two-component quantum liquid we can expect that the quantum vacuum component should also obey the thermodynamic laws, which emerge in any macroscopically large system, relativistic or non-relativistic. This approach allows us to treat the cosmological constant problems. Cosmological constant was introduced by Einstein [1], and was interpreted as the energy density of the quantum vacuum [2, 3]. Astronomical observations (see, e.g., Refs. [4–6]) confirmed the existence of cosmological constant which value corresponds to the energy density of order  $\Lambda_{\text{obs}} \sim E_{\text{obs}}^4$  with the characteristic energy scale  $E_{\text{obs}} \sim 10^{-3}$  eV. However, naive and intuitive theoretical estimation of the vacuum energy density as the zero-point energy of quantum fields suggests that vacuum energy must have the Planck energy scale:  $\sim E_P^4 \sim 10^{120} E_{\text{obs}}^4$ . The huge disagreement between the naive expectations and observations is naturally resolved using the thermodynamics of quantum vacuum discussed in Sections II–V of this review. We shall see that the intuitive estimation for the vacuum energy density as  $\sim E_P^4$  is correct, but the relevant vacuum energy which enters Einstein equations as cosmological constant is somewhat different and its value in the fully equilibrium vacuum is zero.

The second element of the Landau phenomenological approach to macroscopic systems is symmetry. It is in the basis of the modern theory of particle physics – the Standard Model, and its extension to higher energy – the Grand Unification (GUT). The vacuum of Standard Model and GUT obeys the fundamental symmetries which become spontaneously broken at low energy, and are restored when the Planck energy scale is approached from below.

This approach contains another huge disagreement between the naive expectations and observations. It concerns masses of elementary particles. The naive and intuitive estimation tells us that these masses should be on the order of the characteristic energy scale in our vacuum, which is the Planck energy scale,  $M_{\text{theor}} \sim E_P$ . However, the masses of observed particles are many orders of magnitude smaller being below the electroweak energy scale  $M_{\text{obs}} < E_{\text{ew}} \sim 1 \text{ TeV} \sim 10^{-16} E_P$ . This is called the hierarchy problem. There should be a general principle, which could resolve this paradoxes. This is the principle of emergent physics based on the topology in momentum space. This approach supports our intuitive estimation of fermion masses of order  $E_P$ , but this estimation is not valid for such vacua where the massless fermions are topologically protected.

Let us consider fermionic quantum liquid  $^3\text{He}$ . In laboratory we have four different states

of this liquid. These are the normal liquid  $^3\text{He}$ , and three superfluid phases:  $^3\text{He-A}$ ,  $^3\text{He-B}$  and  $^3\text{He-A}_1$ . Only one of them,  $^3\text{He-B}$ , is fully gapped, and for this liquid the intuitive estimation of the gap (analog of mass) in terms of the characteristic energy scale is correct. However, the other liquids are gapless. This gaplessness is protected by the momentum space topology and thus is fundamental: it does not depend much on microscopic physics being robust to the perturbative modification of the interaction between the atoms of the liquid.

### C. Vacuum as topological medium

Topology operates in particular with integer numbers – topological charges – which do not change under small deformation of the system. The conservation of these topological charges protects the Fermi surface and another object in momentum space – the Fermi point – from destruction. They survive when the interaction between the fermions is introduced and modified. When the momentum of a particle approaches the Fermi surface or the Fermi point its energy necessarily vanishes. Thus the topology is the main reason why there are gapless quasiparticles in quantum liquids and (nearly) massless elementary particles in our Universe.

Topology provides the complementary anti-GUT approach in which the ‘fundamental’ symmetry and ‘fundamental’ fields of GUT gradually emerge together with ‘fundamental’ physical laws when the Planck energy scale is approached from above [7, 8]. The emergence of the ‘fundamental’ laws of physics is provided by the general property of topology – robustness to details of the microscopic trans-Planckian physics. As a result, the physical laws which emerge at low energy together with the matter itself are generic. They do not depend much on the details of the trans-Planckian subsystem, being determined by the universality class, which the whole system belongs to.

In this scheme, fermions are primary objects. Approaching the Planck energy scale from above, they are transformed to the Standard Model chiral fermions and give rise to the secondary objects: gauge fields and gravity. Below the Planck scale, the GUT scenario intervenes giving rise to symmetry breaking at low energy. This is accompanied by formation of composite objects, Higgs bosons, and tiny Dirac masses of quark and leptons.

In the GUT scheme, general relativity is assumed to be as fundamental as quantum mechanics, while in the second scheme general relativity is a secondary phenomenon. In the anti-GUT scheme, general relativity is the effective theory describing the dynamics of the effective metric experienced by the effective low-energy fields. It is a side product of quantum field theory or of the quantum mechanics in the vacuum with Fermi point.

Vacua with topologically protected gapless (massless) fermions are representatives of the broader class of topological media. In condensed matter it includes topological insulators (see reviews [9, 10]), topological semimetals (see [11–18]), topological superconductors and superfluids, states which experience quantum Hall effect, and other topologically nontrivial gapless and gapped phases of matter. Topological media have many peculiar properties: topological stability of gap nodes; topologically protected edge states including Majorana fermions; topological quantum phase transitions occurring at  $T = 0$ ; topological quantization of physical parameters including Hall and spin-Hall conductivity; chiral anomaly; topological Chern-Simons and Wess-Zumino actions; etc.

It appears that quantum vacuum of Standard Model is topologically nontrivial both in its massless and massive states. In the massless state the quantum vacuum is topologically

similar to the superfluid  $^3\text{He-A}$  and gapless semimetal. In the massive state the quantum vacuum is topologically similar to the superfluid  $^3\text{He-B}$  and 3+1 dimensional topological insulator. This is discussed in Sections VI–VIII.

## II. QUANTUM VACUUM AS SELF-SUSTAINED MEDIUM

### A. Vacuum energy and cosmological constant

There is a huge contribution to the vacuum energy density, which comes from the ultraviolet (Planckian) degrees of freedom and is of order  $E_{\text{P}}^4 \approx (10^{28} \text{ eV})^4$ . The observed cosmological is smaller by many orders of magnitude and corresponds to the energy density of the vacuum  $\rho_{\text{vac}} \sim (10^{-3} \text{ eV})^4$ . In general relativity, the cosmological constant is arbitrary constant, and thus its smallness requires fine-tuning. If gravitation would be a truly fundamental interaction, it would be hard to understand why the energies stored in the quantum vacuum would not gravitate at all [19]. If, however, gravitation would be only a low-energy effective interaction, it could be that the corresponding gravitons as quasiparticles do not feel *all* microscopic degrees of freedom (gravitons would be analogous to small-amplitude waves at the surface of the ocean) and that the gravitating effect of the vacuum energy density would be effectively *tuned away* and cosmological constant would be naturally small or zero [8, 20].

### B. Variables for Lorentz invariant vacuum

A particular mechanism of nullification of the relevant vacuum energy works for such vacua which have the property of a *self-sustained medium* [21–26]. A self-sustained vacuum is a medium with a definite macroscopic volume even in the absence of an environment. A condensed matter example is a droplet of quantum liquid at zero temperature in empty space. The observed near-zero value of the cosmological constant compared to Planck-scale values suggests that the quantum vacuum of our universe belongs to this class of systems. As any medium of this kind, the equilibrium vacuum would be homogeneous and extensive. The homogeneity assumption is indeed supported by the observed flatness and smoothness of our universe [27–29]. The implication is that the energy of the equilibrium quantum vacuum would be proportional to the volume considered.

Usually, a self-sustained medium is characterized by an *extensive conserved quantity* whose total value determines the actual volume of the system [30, 31]. The quantum liquid at  $T = 0$  is a self sustained system because of the conservation law for the particle number  $N$ , and its state is characterized by the particle density  $n$  which acquires a non-zero value  $n = n_0$  in the equilibrium ground state. As distinct from condensed matter systems, the quantum vacuum of our Universe is a relativistic invariant system. The Lorentz invariance of the vacuum imposes strong constraints on the possible form this variable can take. One must find the relativistic analog of the particle density  $n$ . An example of a possible vacuum variable is a symmetric tensor  $q^{\mu\nu}$ , which in a homogeneous vacuum is proportional to the metric tensor

$$q^{\mu\nu} = q g^{\mu\nu}. \quad (2.1)$$

This variable satisfies the Lorentz invariance of the vacuum. Another example is the 4-tensor  $q^{\mu\nu\alpha\beta}$ , which in a homogeneous vacuum is proportional either to the fully antisymmetric Levi-Civita tensor:

$$q^{\mu\nu\alpha\beta} = q e^{\mu\nu\alpha\beta}, \quad (2.2)$$

or to the product of metric tensors such as:

$$q^{\mu\nu\alpha\beta} = q (g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}). \quad (2.3)$$

Scalar field is also the Lorentz invariant variable, but it does not satisfy another necessary condition of the self sustained system: the vacuum variable  $q$  must obey some kind of the conservation law. Below we consider some examples satisfying the two conditions: Lorentz invariance of the perfect vacuum state and the conservation law.

### C. Yang-Mills chiral condensate as example

Let us first consider as an example the chiral condensate of gauge fields. It can be the gluonic condensate in QCD [32, 33], or any other condensate of Yang-Mills fields, if it is Lorentz invariant. We assume that the Savvidy vacuum [34] is absent, i.e. the vacuum expectation value of the color magnetic field is zero (we shall omit color indices):

$$\langle F_{\alpha\beta} \rangle = 0, \quad (2.4)$$

while the vacuum expectation value of the quadratic form is nonzero:

$$\langle F_{\alpha\beta} F_{\mu\nu} \rangle = \frac{q}{24} \sqrt{-g} e_{\alpha\beta\mu\nu}. \quad (2.5)$$

Here  $q$  is the anomaly-driven topological condensate (see e.g. [35]):

$$q = \left\langle \tilde{F}^{\mu\nu} F_{\mu\nu} \right\rangle = \frac{1}{\sqrt{-g}} e^{\alpha\beta\mu\nu} \langle F_{\alpha\beta} F_{\mu\nu} \rangle, \quad (2.6)$$

In the homogeneous static vacuum state, the  $q$ -condensate violates the  $P$  and  $T$  symmetries of the vacuum, but it conserves the combined symmetry  $PT$  symmetry.

#### 1. Cosmological term

Let us choose the vacuum action in the form

$$S_q = \int d^4x \sqrt{-g} \epsilon(q), \quad (2.7)$$

with  $q$  given by (2.6). The energy-momentum tensor of the vacuum field  $q$  is obtained by variation of the action over  $g^{\mu\nu}$ :

$$T_{\mu\nu}^q = -\frac{2}{\sqrt{-g}} \frac{\delta S_q}{\delta g^{\mu\nu}} = \epsilon(q) g_{\mu\nu} - 2 \frac{\partial \epsilon}{\partial q} \frac{\partial q}{\partial g^{\mu\nu}}. \quad (2.8)$$

Using (2.5) and (2.6) one obtains

$$\frac{\partial q}{\partial g^{\mu\nu}} = \frac{1}{2} q g_{\mu\nu} . \quad (2.9)$$

and thus

$$T_{\mu\nu}^q = g_{\mu\nu} \rho_{\text{vac}}(q) \quad , \quad \rho_{\text{vac}}(q) = \epsilon(q) - q \frac{\partial \epsilon}{\partial q} . \quad (2.10)$$

In Einstein equations this energy momentum tensor plays the role of the cosmological term:

$$T_{\mu\nu}^q = \Lambda g_{\mu\nu} \quad , \quad \Lambda = \rho_{\text{vac}}(q) = \epsilon(q) - q \frac{\partial \epsilon}{\partial q} . \quad (2.11)$$

It is important that the cosmological constant is given not by the vacuum energy as is usually assumed, but by thermodynamic potential  $\rho_{\text{vac}} = \epsilon(q) - \mu q$ , where  $\mu$  is thermodynamically conjugate to  $q$  variable,  $\mu = d\epsilon/dq$ . Below, when we consider dynamics we shall see that this fact reflects the conservation of the variable  $q$ .

The crucial difference between the vacuum energy  $\epsilon(q)$  and thermodynamic potential  $\rho_{\text{vac}} = \epsilon(q) - \mu q$  is revealed when we consider the corresponding quantities in the ground state of quantum liquids, the energy density of the liquid  $\epsilon(n)$  and the density of the grand canonical energy,  $\epsilon(n) - \mu n$ , which enters macroscopic thermodynamics due to conservation of particle number. The first one,  $\epsilon(n)$ , has the value dictated by atomic physics, which is equivalent to  $E_{\text{P}}^4$  in the quantum vacuum. On the contrary, the second one equals minus pressure,  $\epsilon(n) - \mu n = -P$ , according to the Gibbs-Duhem thermodynamic relation at  $T = 0$ . Thus its value is dictated not by the microscopic physics, but by external conditions. In the absence of environment, the external pressure is zero, and the value of  $\epsilon(n) - \mu n$  in a fully equilibrium ground state of the liquid is zero. This is valid for any self-sustained macroscopic system, including the self-sustained quantum vacuum, which suggests the natural solution of the main cosmological constant problem.

## 2. Conservation law for $q$

Equation for  $q$  in flat space can be obtained from Maxwell equation, which in turn is obtained by variation of the action over the gauge field  $A_\mu$ :

$$\nabla_\mu \left( \frac{\partial \epsilon}{\partial q} \tilde{F}^{\mu\nu} \right) = 0 , \quad (2.12)$$

where  $\nabla_\mu$  is the covariant derivative. Since  $\nabla_\mu \tilde{F}^{\mu\nu} = 0$ , equation (4.3) is reduced to

$$\nabla_\mu \left( \frac{\partial \epsilon}{\partial q} \right) = 0 . \quad (2.13)$$

The solution of this equation is

$$\frac{\partial \epsilon}{\partial q} = \mu , \quad (2.14)$$

where  $\mu$  is integration constant. In thermodynamics, this  $\mu$  will play the role of the chemical potential, which is thermodynamically conjugate to  $q$ . This demonstrates that  $q$  obeys the conservation law and thus can be the proper variable for description the self-sustained vacuum.



#### D. 4-form field as example

Another example of the vacuum variable appropriate for the self-sustained vacuum is given by the four-form field strength [36–44], which is expressed in terms of  $q$  in the following way:

$$F_{\alpha\beta\gamma\delta} \equiv q e_{\alpha\beta\gamma\delta} \sqrt{-\det g} = \nabla_{[\alpha} A_{\beta\gamma\delta]}, \quad (2.15a)$$

$$q^2 = -\frac{1}{24} F_{\alpha\beta\gamma\delta} F^{\alpha\beta\gamma\delta}, \quad (2.15b)$$

where  $e_{\alpha\beta\gamma\delta}$  the Levi–Civita tensor density; and the square bracket around spacetime indices complete anti-symmetrization.

Originally the quadratic action has been used for this field [36, 37], which corresponds to the special case of (2.7) with  $\epsilon(q) = \frac{1}{2} q^2$ . For the general  $\epsilon(q)$  one obtains the Maxwell equation

$$\nabla_\alpha \left( \sqrt{-\det g} \frac{F^{\alpha\beta\gamma\delta}}{q} \frac{\partial \epsilon(q)}{\partial q} \right) = 0. \quad (2.16)$$

Using (2.15a) the Maxwell equation is reduced to

$$\nabla_\alpha \left( \frac{\partial \epsilon(q)}{\partial q} \right) = 0. \quad (2.17)$$

The first integral of (2.17) with integration constant  $\mu$  gives again Eq.(2.14), which reflects the conservation law for  $q$ .

Variation of the action over  $g^{\mu\nu}$  gives again the cosmological constant (2.11) with  $\Lambda = \rho_{\text{vac}} = \epsilon(q) - \mu q$ . This demonstrates the universality of the macroscopic description of the self-sustained vacuum: description of the quantum vacuum in terms of  $q$  does not depend on the microscopic details of the vacuum and on the nature of the vacuum variable.

#### E. Aether field as example

Another example of the vacuum variable  $q$  may be through a four-vector field  $u^\mu(x)$ . This vector field could be the four-dimensional analog of the concept of shift in the deformation theory of crystals. (Deformation theory can be described in terms of a metric field, with the role of torsion and curvature fields played by dislocations and disclinations, respectively; see, e.g., Ref. [45] for a review.) A realization of  $u^\mu$  could be also a 4-velocity field entering the description of the structure of spacetime. It is the 4-velocity of “aether” [46–49].

The nonzero value of the 4-vector in the vacuum violates the Lorentz invariance of the vacuum. To restore this invariance one may assume that  $u^\mu(x)$  is not an observable variable, instead the observables are its covariant derivatives  $\nabla_\nu u^\mu \equiv u^\mu_\nu$ . This means that the action does not depend on  $u^\mu$  explicitly but only depends on  $u^\mu_\nu$ :

$$S = \int_{\mathbb{R}^4} d^4x \epsilon(u^\mu_\nu), \quad (2.18)$$

with an energy density containing even powers of  $u^\mu_\nu$ :

$$\epsilon(u^\mu_\nu) = K + K_{\mu\nu}^{\alpha\beta} u^\mu_\alpha u^\nu_\beta + K_{\mu\nu\rho\sigma}^{\alpha\beta\gamma\delta} u^\mu_\alpha u^\nu_\beta u^\rho_\gamma u^\sigma_\delta + \dots. \quad (2.19)$$

According to the imposed conditions, the tensors  $K_{\mu\nu}^{\alpha\beta}$  and  $K_{\mu\nu\rho\sigma}^{\alpha\beta\gamma\delta}$  depend only on  $g_{\mu\nu}$  or  $g^{\mu\nu}$  and the same holds for the other  $K$ -like tensors in the ellipsis of (2.19). In particular, the tensor  $K_{\mu\nu}^{\alpha\beta}$  of the quadratic term in (2.19) has the following form in the notation of Ref. [46]:

$$K_{\mu\nu}^{\alpha\beta} = c_1 g^{\alpha\beta} g_{\mu\nu} + c_2 \delta_\mu^\alpha \delta_\nu^\beta + c_3 \delta_\nu^\alpha \delta_\mu^\beta, \quad (2.20)$$

for real constants  $c_n$ . Distinct from the original aether theory in Ref. [46], the tensor (2.20) does not contain a term  $c_4 u^\alpha u^\beta g_{\mu\nu}$ , as such a term would depend explicitly on  $u^\mu$  and contradict the Lorentz invariance of the quantum vacuum.

The equation of motion for  $u^\mu$  in flat space,

$$\nabla_\nu \frac{\partial \epsilon}{\partial u_\nu^\mu} = 0, \quad (2.21)$$

has the Lorentz invariant solution expected for a vacuum-variable  $q$ -type field:

$$u_{\mu\nu}^q = q g_{\mu\nu}, \quad q = \text{constant}. \quad (2.22)$$

With this solution, the energy density in the action (2.18) is simply  $\epsilon(q)$  in terms of contracted coefficients  $K$ ,  $K_{\mu\nu}^{\mu\nu}$ , and  $K_{\mu\nu\rho\sigma}^{\mu\nu\rho\sigma}$  from (2.19). However, just as for previous examples, the energy-momentum tensor of the vacuum field obtained by variation over  $g^{\mu\nu}$  and evaluated for solution (2.22) is expressed again in terms of the thermodynamic potential:

$$T_{\mu\nu}^q = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = g_{\mu\nu} \left( \epsilon(q) - q \frac{d\epsilon(q)}{dq} \right) = \rho_{\text{vac}}(q) g_{\mu\nu}, \quad (2.23)$$

which corresponds to cosmological constant in Einstein's gravitational field equations.

### III. THERMODYNAMICS OF QUANTUM VACUUM

#### A. Liquid-like quantum vacuum

The zeroth order term  $K$  in (2.19) corresponds to a “bare” cosmological constant which can be considered as cosmological constant in the “empty” vacuum – vacuum with  $q = 0$ :

$$\Lambda_{\text{bare}} = \epsilon(q = 0). \quad (3.1)$$

The nonzero value  $q = q_0$  in the self-sustained vacuum does *not* violate Lorentz symmetry but leads to compensation of the bare cosmological constant  $\Lambda_{\text{bare}}$  in the equilibrium vacuum. This illustrates the important difference between the two states of vacua. The quantum vacuum with  $q = 0$  can exist only with external pressure  $P = -\Lambda_{\text{bare}}$ . By analogy with condensed-matter physics, this kind of quantum vacuum may be called “gas-like” (Fig. 1). The quantum vacuum with nonzero  $q$  is self-sustained: it can be stable at  $P = 0$ , provided that a stable nonzero solution of equation  $\epsilon(q) - q d\epsilon/dq = 0$  exists. This kind of quantum vacuum may then be called “liquid-like”.

The universal behavior of the self-sustained vacuum in equilibrium suggests that it obeys the same thermodynamic laws as any other self-sustained macroscopic system described by the conserved quantity  $q$ , such as quantum liquid. In other words, vacuum can be considered

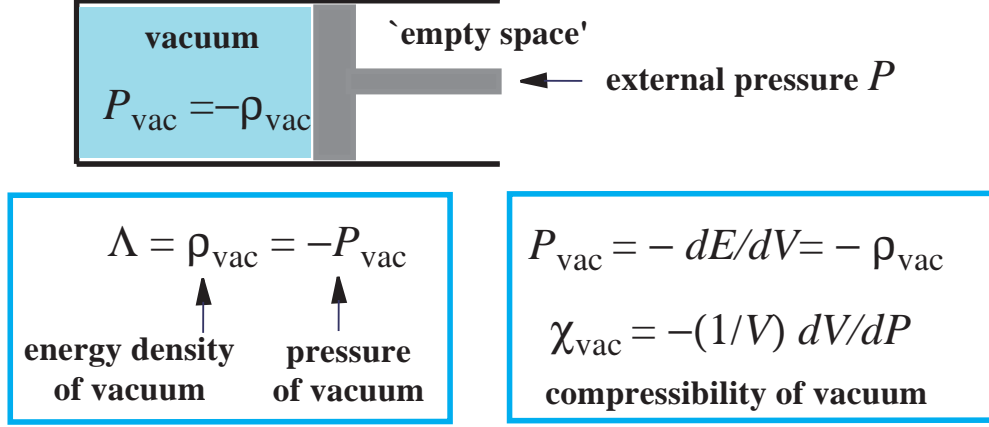


FIG. 1: Vacuum as a medium obeying macroscopic thermodynamic laws. Relativistic vacuum possesses energy density, pressure and compressibility but has no momentum. In equilibrium, the vacuum pressure  $P_{\text{vac}}$  equals the external pressure  $P$  acting from the environment. The “gas-like” vacuum may exist only under external pressure. The “liquid-like” vacuum is self-sustained: it can be stable in the absence of external pressure. The thermodynamic energy density of the vacuum  $\rho_{\text{vac}}$  which enters the vacuum equation of state  $\rho_{\text{vac}} = -P_{\text{vac}}$  does not coincide with the microscopic vacuum energy  $\epsilon$ . While the natural value of  $\epsilon$  is determined by the Planck scale,  $\epsilon \sim E_{\text{P}}^4$ , the natural value of the macroscopic quantity  $\rho_{\text{vac}}$  is zero for the self-sustained vacuum which may exist in the absence of environment, i.e. at  $P = 0$ . This may explain why the present cosmological constant  $\Lambda = \rho_{\text{vac}}$  is small.

as a special quantum liquid which is Lorentz invariant in its ground state. This liquid is characterized by the Lorentz invariant “charge” density  $q$  – an analog of particle density  $n$  in non-relativistic quantum liquids.

Let us consider a large portion of such vacuum liquid under external pressure  $P$  [21]. The volume  $V$  of quantum vacuum is variable, but its total “charge”  $Q(t) \equiv \int d^3r q(\mathbf{r}, t)$  must be conserved,  $dQ/dt = 0$ . The energy of this portion of quantum vacuum at fixed total “charge”  $Q = qV$  is then given by the thermodynamic potential

$$W = E + PV = \int d^3r \epsilon(Q/V) + PV, \quad (3.2)$$

where  $\epsilon(q)$  is the energy density in terms of charge density  $q$ . As the volume of the system is a free parameter, the equilibrium state of the system is obtained by variation over the volume  $V$ :

$$\frac{dW}{dV} = 0, \quad (3.3)$$

This gives an integrated form of the Gibbs–Duhem equation for the vacuum pressure:

$$P_{\text{vac}} = -\epsilon(q) + q \frac{d\epsilon(q)}{dq} = -\rho_{\text{vac}}(q), \quad (3.4)$$

whose solution determines the equilibrium value  $q = q(P)$  and the corresponding volume  $V = (P, Q) = Q/q(P)$ .

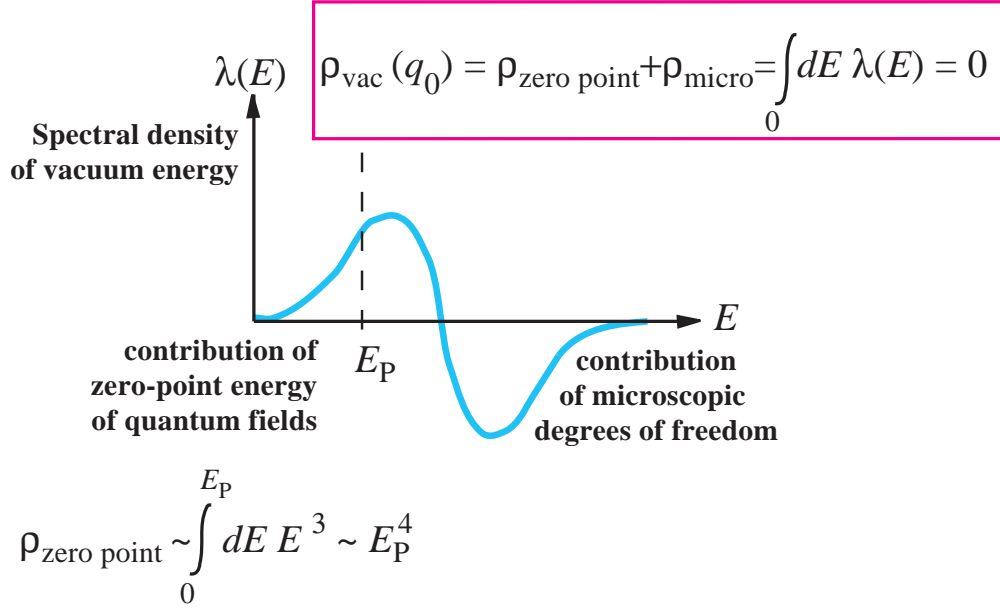


FIG. 2: Contribution of different energy scales into the macroscopic energy of the self-sustained system at  $T = 0$ . Zero point energy of the effective bosonic and fermionic quantum fields gives rise to the diverging contribution to the energy of the system. In quantum vacuum it is of order of  $E_P^4$ . In equilibrium this contribution is compensated without fine-tuning by microscopic degrees of freedom of the system (by trans-Planckian degrees of quantum vacuum correspondingly).

## B. Macroscopic energy of quantum vacuum

Since the vacuum energy density is the vacuum pressure with minus sign, equation (3.4) suggests that the relevant vacuum energy, which is revealed in thermodynamics and dynamics of the low-energy Universe, is:

$$\rho_{\text{vac}}(q) = \epsilon(q) - q \frac{d\epsilon(q)}{dq} . \quad (3.5)$$

This is confirmed by Eqs. (2.11) and (2.23) for energy-momentum tensor of the self-sustained vacuum, which demonstrates that it is  $\rho_{\text{vac}}(q)$  rather than  $\epsilon(q)$ , which enters the equation of state for the vacuum and thus corresponds to the cosmological constant:

$$\Lambda = \rho_{\text{vac}} = -P_{\text{vac}} . \quad (3.6)$$

While the energy of microscopic quantity  $q$  is determined by the Planck scale,  $\epsilon(q_0) \sim E_P^4$ , the relevant vacuum energy which sources the effective gravity is determined by a macroscopic quantity – the external pressure. In the absence of an environment, i.e. at zero external pressure,  $P = 0$ , one obtains that the pressure of pure and equilibrium vacuum is exactly zero:

$$\Lambda = -P_{\text{vac}} = -P = 0 . \quad (3.7)$$

Equation  $\rho_{\text{vac}}(q) = 0$  determines the equilibrium value  $q_0$  of the equilibrium self-sustained vacuum. Thus from the thermodynamic arguments it follows that for any effective theory of gravity the natural value of  $\Lambda$  is zero in equilibrium vacuum.

This result does not depend on the microscopic structure of the vacuum from which gravity emerges, and is actually the final result of the renormalization dictated by macroscopic physics. In the self-sustained quantum liquid the large contribution of zero-point energy of phonon field is naturally compensated by microscopic (atomic) degrees of freedom of quantum liquid. In the same manner, the huge contribution of zero-point energy of macroscopic fields to the vacuum energy  $\rho_{\text{vac}}$  is naturally compensated by microscopic degrees of the self-sustained quantum vacuum: the vacuum variable  $q$  is adjusted automatically to nullify the macroscopic vacuum energy,  $\rho_{\text{vac}}(q_0) = \rho_{\text{zero point}} + \rho_{\text{microscopic}} = 0$ . The actual spectrum of the vacuum energy density (meaning the different contributions to  $\epsilon$  from different energy scales) is not important for the cancellation mechanism, because it is dictated by thermodynamics. The particular example of the spectrum of the vacuum energy density is shown in Fig. 2, where the positive energy of the quantum vacuum, which comes from the zero-point energy of bosonic fields, is compensated by negative contribution from trans-Planckian degrees of freedom [50].

Using the quantum-liquid counterpart of the self-sustained quantum vacuum as example, one may predict the behavior of the vacuum after cosmological phase transition, when  $\Lambda$  is kicked from its zero value. The vacuum will readjust itself to a new equilibrium state with new  $q_0$  so that  $\Lambda$  will again approach its equilibrium zero value [21]. The process of relaxation of the system to the equilibrium state depends on details of dynamics of the vacuum variable  $q$  and its interaction with matter fields, and later on we shall consider some examples of dynamical relaxation of  $\Lambda$ .

### C. Compressibility of the vacuum

Using the standard definition of the inverse of the isothermal compressibility,  $\chi^{-1} \equiv -V dP/dV$  (Fig. 1), one obtains the compressibility of the vacuum by varying Eq.(3.4) at fixed  $Q = qV$  [21]:

$$\chi_{\text{vac}}^{-1} \equiv -V \frac{dP_{\text{vac}}}{dV} = \left[ q^2 \frac{d^2 \epsilon(q)}{dq^2} \right]_{q=q_0} > 0. \quad (3.8)$$

A positive value of the vacuum compressibility is a necessary condition for the stability of the vacuum. It is, in fact, the stability of the vacuum, which is at the origin of the nullification of the cosmological constant in the absence of an external environment.

From the low-energy point of view, the compressibility of the vacuum  $\chi_{\text{vac}}$  is as fundamental physical constant as the Newton constant  $G_N = G(q = q_0)$ . It enters equations describing the response of the quantum vacuum to different perturbations. While the natural value of the macroscopic quantity  $P_{\text{vac}}$  (and  $\rho_{\text{vac}}$ ) is zero, the natural values of the parameters  $G(q = q_0)$  and  $\chi_{\text{vac}}(q = q_0)$  are determined by the Planck physics and are expected to be of order  $1/E_{\text{P}}^2$  and  $1/E_{\text{P}}^4$  correspondingly.

### D. Thermal fluctuations of $\Lambda$ and the volume of Universe

The compressibility of the vacuum  $\chi_{\text{vac}}$ , though not measurable at the moment, can be used for estimation of the lower limit for the volume  $V$  of the Universe. This estimation follows from the upper limit for thermal fluctuations of cosmological constant [51]. The

mean square of thermal fluctuations of  $\Lambda$  equals the mean square of thermal fluctuations of the vacuum pressure, which in turn is determined by thermodynamic equation [30]:

$$\langle(\Delta\Lambda)^2\rangle = \langle(\Delta P)^2\rangle = \frac{T}{V\chi_{\text{vac}}} . \quad (3.9)$$

Typical fluctuations of the cosmological constant  $\Lambda$  should not exceed the observed value:  $\langle(\Delta\Lambda)^2\rangle < \Lambda_{\text{obs}}^2$ . Let us assume, for example, that the temperature of the Universe is determined by the temperature  $T_{\text{CMB}}$  of the cosmic microwave background radiation. Then, using our estimate for vacuum compressibility  $\chi_{\text{vac}}^{-1} \sim E_{\text{P}}^4$ , one obtains that the volume  $V$  of our Universe highly exceeds the Hubble volume  $V_H = R_H^3$  – the volume of visible Universe inside the present cosmological horizon:

$$V > \frac{T_{\text{CMB}}}{\chi_{\text{vac}}\Lambda_{\text{obs}}^2} \sim 10^{28}V_H . \quad (3.10)$$

This demonstrates that the real volume of the Universe is certainly not limited by the present cosmological horizon.

## IV. DYNAMICS OF QUANTUM VACUUM

### A. Action

In section II a special quantity, the vacuum “charge”  $q$ , was introduced to describe the statics and thermodynamics of the self-sustained quantum vacuum. Now we can extend this approach to the dynamics of the vacuum charge. We expect to find some universal features of the vacuum dynamics, using several realizations of this vacuum variable. We start with the 4-form field strength [36–44] expressed in terms of  $q$ . The low-energy effective action takes the following general form:

$$S = - \int_{\mathbb{R}^4} d^4x \sqrt{|g|} \left( \frac{R}{16\pi G(q)} + \epsilon(q) + \mathcal{L}^{\text{M}}(q, \psi) \right) , \quad (4.1a)$$

$$q^2 \equiv -\frac{1}{24} F_{\kappa\lambda\mu\nu} F^{\kappa\lambda\mu\nu} , \quad F_{\kappa\lambda\mu\nu} \equiv \nabla_{[\kappa} A_{\lambda\mu\nu]} , \quad (4.1b)$$

$$F_{\kappa\lambda\mu\nu} = q\sqrt{|g|} e_{\kappa\lambda\mu\nu} , \quad F^{\kappa\lambda\mu\nu} = q e^{\kappa\lambda\mu\nu} / \sqrt{|g|} . \quad (4.1c)$$

where  $R$  denotes the Ricci curvature scalar; and  $\mathcal{L}^{\text{M}}$  is matter action. Throughout, we use the same conventions as in Ref. [52], in particular, those for the Riemann curvature tensor and the metric signature  $(-+++)$ .

The vacuum energy density  $\epsilon$  in (4.1a) depends on the vacuum variable  $q$  which in turn is expressed via the 3-form field  $A_{\lambda\mu\nu}$  and metric field  $g_{\mu\nu}$  in (4.1b). The field  $\psi$  combines all the matter fields of the Standard Model. All possible constant terms in matter action (which includes the zero-point energies from the Standard Model fields) are absorbed in the vacuum energy  $\epsilon(q)$ .

Since  $q$  describes the state of the vacuum, the parameters of the effective action – the Newton constant  $G$  and parameters which enter the matter action – must depend on  $q$ . This dependence results in particular in the interaction between the matter fields and the

vacuum. There are different sources of this interaction. For example, in the gauge field sector of Standard Model, the running coupling contains the ultraviolet cut-off and thus depends on  $q$ :

$$\mathcal{L}^{\mathbf{G},\mathbf{q}} = \gamma(q) F^{\mu\nu} F_{\mu\nu}, \quad (4.2)$$

where  $F_{\mu\nu}$  is the field strength of the particular gauge field (we omitted the color indices). In the fermionic sector,  $q$  should enter parameters of the Yukawa interaction and fermion masses.

## B. Vacuum dynamics

The variation of the action (4.1a) over the three-form gauge field  $A$  gives the generalized Maxwell equations for  $F$ -field,

$$\nabla_\nu \left( \sqrt{|g|} \frac{F^{\kappa\lambda\mu\nu}}{q} \left( \frac{d\epsilon(q)}{dq} + \frac{R}{16\pi} \frac{dG^{-1}(q)}{dq} + \frac{d\mathcal{L}^{\mathbf{M}}(q)}{dq} \right) \right) = 0. \quad (4.3)$$

Using (4.1c) for  $F^{\kappa\lambda\mu\nu}$ , we find that the solutions of Maxwell equations (4.3) are still determined by the integration constant  $\mu$

$$\frac{d\epsilon(q)}{dq} + \frac{R}{16\pi} \frac{dG^{-1}(q)}{dq} + \frac{d\mathcal{L}^{\mathbf{M}}(q)}{dq} = \mu. \quad (4.4)$$

## C. Generalized Einstein equations

The variation over the metric  $g^{\mu\nu}$  gives the generalized Einstein equations,

$$\begin{aligned} & \frac{1}{8\pi G(q)} \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) + \frac{1}{16\pi} q \frac{dG^{-1}(q)}{dq} R g_{\mu\nu} \\ & + \frac{1}{8\pi} \left( \nabla_\mu \nabla_\nu G^{-1}(q) - g_{\mu\nu} \square G^{-1}(q) \right) - \left( \epsilon(q) - q \frac{d\epsilon(q)}{dq} \right) g_{\mu\nu} \\ & + q \frac{\partial \mathcal{L}^{\mathbf{M}}}{\partial q} g_{\mu\nu} + T_{\mu\nu}^{\mathbf{M}} = 0, \end{aligned} \quad (4.5)$$

where  $\square$  is the invariant d'Alembertian; and  $T_{\mu\nu}^{\mathbf{M}}$  is the energy-momentum tensor of the matter fields, obtained by variation over  $g^{\mu\nu}$  at constant  $q$ , i.e. without variation over  $g^{\mu\nu}$ , which enters  $q$ .

Eliminating  $dG^{-1}/dq$  and  $\partial \mathcal{L}^{\mathbf{M}}/\partial q$  from (4.5) by use of (4.4), the generalized Einstein equations become

$$\frac{1}{8\pi G(q)} \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) + \frac{1}{8\pi} \left( \nabla_\mu \nabla_\nu G^{-1}(q) - g_{\mu\nu} \square G^{-1}(q) \right) - \rho_{\text{vac}} g_{\mu\nu} + T_{\mu\nu}^{\mathbf{M}} = 0, \quad (4.6)$$

where

$$\rho_{\text{vac}} = \epsilon(q) - \mu q. \quad (4.7)$$

For the special case when the dependence of the Newton constant and matter action on  $q$  is ignored, (4.6) reduces to the standard Einstein equation of general relativity with the constant cosmological constant  $\Lambda = \rho_{\text{vac}}$ .

#### D. Minkowski-type solution and Weinberg problem

Among different solutions of equations (4.3) and (4.5) there is the solution corresponding to perfect equilibrium Minkowski vacuum without matter. It is characterized by the constant in space and time values  $q = q_0$  and  $\mu = \mu_0$  obeying the following two conditions:

$$\left[ \frac{d\epsilon(q)}{dq} - \mu \right]_{\mu=\mu_0, q=q_0} = 0, \quad (4.8a)$$

$$\left[ \epsilon(q) - \mu q \right]_{\mu=\mu_0, q=q_0} = 0. \quad (4.8b)$$

The two conditions (4.8a)–(4.8b) can be combined into a *single* equilibrium condition for  $q_0$ :

$$\Lambda_0 \equiv \left[ \epsilon(q) - q \frac{d\epsilon(q)}{dq} \right]_{q=q_0} = 0, \quad (4.9)$$

with the *derived* quantity

$$\mu_0 = \left[ \frac{d\epsilon(q)}{dq} \right]_{q=q_0}. \quad (4.10)$$

In order for the Minkowski vacuum to be stable, there is the further condition:  $\chi(q_0) > 0$  where  $\chi$  corresponds to the isothermal vacuum compressibility (3.8) [21]. In this equilibrium vacuum the gravitational constant  $G(q_0)$  can be identified with Newton's constant  $G_N$ .

Let us compare the conditions for the equilibrium self-sustained vacuum, (4.9) and (4.10), with the two conditions suggested by Weinberg, who used the fundamental scalar field  $\phi$  for the description of the vacuum. In this description there are two constant-field equilibrium conditions for Minkowski vacuum,  $\partial\mathcal{L}/\partial g_{\alpha\beta} = 0$  and  $\partial\mathcal{L}/\partial\phi = 0$ , see Eqs. (6.2) and (6.3) in [52]. These two conditions turn out to be inconsistent, unless the potential term in  $\mathcal{L}(\phi)$  is fine-tuned (see also Sec. 2 of Ref. [53]). In other words, the Minkowski vacuum solution may exist only for the fine-tuned action. This is the Weinberg formulation of the cosmological constant problem.

The self-sustained vacuum naturally bypasses this problem [26]. Equation  $\partial\mathcal{L}/\partial g_{\alpha\beta} = 0$  corresponds to the equation (4.9). However, the equation  $\partial\mathcal{L}/\partial\phi = 0$  is *relaxed* in the  $q$ -theory of self-sustained vacuum. Instead of the condition  $\partial\mathcal{L}/\partial q = 0$ , the conditions are  $\nabla_\alpha(\partial\mathcal{L}/\partial q) = 0$ , which allow for having  $\partial\mathcal{L}/\partial q = \mu$  with an *arbitrary* constant  $\mu$ . This is the crucial difference between a fundamental scalar field  $\phi$  and the variable  $q$  describing the self-sustained vacuum. As a result, the equilibrium conditions for  $g_{\alpha\beta}$  and  $q$  can be consistent without fine-tuning of the original action. For Minkowski vacuum to exist only one condition (4.9) must be satisfied. In other words, the Minkowski vacuum solution exists for arbitrary action provided that solution of equation (4.9) exists.

#### E. Multiple Minkowski vacua

It is instructive to illustrate this using a concrete example. The particular choice for the vacuum energy density function is considered in [26]:

$$\epsilon(q) = \Lambda_{\text{bare}} + (1/2) (E_P)^4 \sin [q^2/(E_P)^4]. \quad (4.11)$$



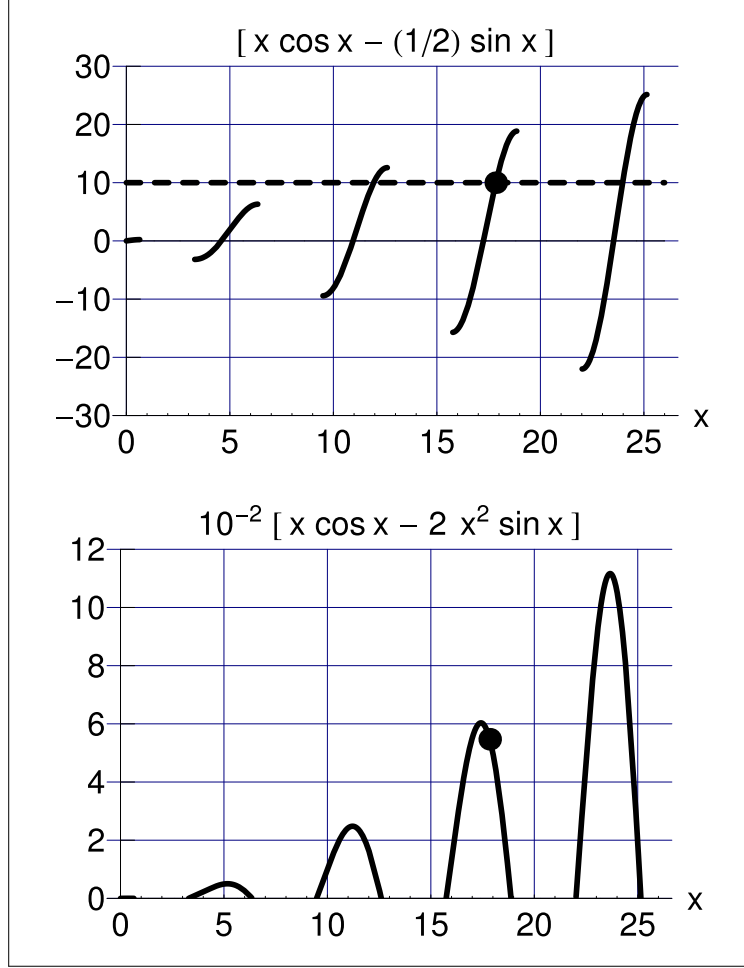


FIG. 3: A set of Minkowski equilibrium vacua emerging for a particular choice of the vacuum energy density function in (4.11). In each vacuum the huge bare cosmological constant  $\Lambda_{\text{bare}} \sim E_{\text{P}}^4$  is compensated by the  $q$ -field. The curves of the top panel show the left-hand side of (4.12a) for those values of  $x \equiv q^2/E_{\text{P}}^4$  that obey the stability condition (4.12b). The curves of the bottom panel show the corresponding positive segments of the inverse of the dimensionless vacuum compressibility  $\chi E_{\text{P}}^4$ . Minkowski-type vacua are obtained at the intersection points of the curve of the top panel with a horizontal line at the value  $\lambda_{\text{bare}} \equiv \Lambda_{\text{bare}}/E_{\text{P}}^4$  [for example, the dashed line at  $\lambda_{\text{bare}} = 10$  gives the value  $x_0 \approx 17.8$  corresponding to the heavy dot in the top panel]. Each such vacuum is characterized, in part, by the corresponding value of the inverse vacuum compressibility from the bottom panel shown by the heavy dot for the case chosen in the top panel]. Minkowski vacua with positive compressibility are stable and become attractors in a dynamical context (cf. next Section).

It contains the higher-order terms in addition to the standard quadratic term  $\frac{1}{2}q^2$ . With (4.11), the expressions for the equilibrium condition (4.9) and the stability condition (3.8)

become

$$x \cos x - (1/2) \sin x = \lambda_{\text{bare}}, \quad (4.12a)$$

$$\chi^{-1} E_{\text{P}}^{-4} = x \cos x - 2x^2 \sin x > 0, \quad (4.12b)$$

where dimensionless quantities  $x \equiv q^2/E_{\text{P}}^4$  and  $\lambda_{\text{bare}} \equiv \Lambda_{\text{bare}}/E_{\text{P}}^4$  are introduced. A straightforward graphical analysis (Fig. 3) shows that, for any  $\lambda_{\text{bare}} \in \mathbb{R}$ , there are infinitely many equilibrium states of quantum vacuum, i.e. infinitely many values  $q_0 \in \mathbb{R}$  which obey both (4.12a) and (4.12b). Each of these vacua has its own values of the Newton constant  $G(q_0)$  and Standard Model parameters. But all these vacua have zero cosmological constant: the Planck-scale bare cosmological constant  $\Lambda_{\text{bare}}$  is compensated by the  $q$  field in any equilibrium vacuum. The top panel of Fig. 3 shows that the  $q$  values on the one segment singled-out by the heavy dot already allow for a complete cancellation of *any*  $\Lambda_{\text{bare}}$  value between  $-15 E_{\text{P}}^4$  and  $+18 E_{\text{P}}^4$ .

## V. COSMOLOGY AS APPROACH TO EQUILIBRIUM

### A. Energy exchange between vacuum and gravity+matter

In the curved Universe and/or in the presence of matter,  $q$  becomes space-time dependent due to interaction with gravity and matter (see (4.4)). As a result the vacuum energy can be transferred to the energy of gravitational field and/or to the energy of matter fields. This also means that the energy of matter is not conserved. The energy-momentum tensor of matter  $T_{\mu\nu}^{\text{M}}$ , which enters the generalized Einstein equations (4.6), is determined by variation over  $g^{\mu\nu}$  at constant  $q$ . That is why it is not conserved:

$$\nabla_\nu T^{\text{M}\mu\nu} = -\frac{\partial \mathcal{L}^{\text{M}}}{\partial q} \nabla_\mu q. \quad (5.1)$$

The matter energy can be transferred to the vacuum energy due to interaction with  $q$ -field. Using (4.4) and the equation (4.7) for cosmological constant one obtains that the vacuum energy is transferred both to gravity and matter with the rate:

$$\nabla_\mu \Lambda \equiv \nabla_\mu \rho_{\text{vac}} = \left( \frac{d\epsilon(q)}{dq} - \mu \right) \nabla_\mu q = -\frac{R}{16\pi} \frac{dG^{-1}(q)}{dq} \nabla_\mu q + \nabla_\nu T^{\text{M}\mu\nu}. \quad (5.2)$$

The energy exchange between the vacuum and gravity+matter allows for the relaxation of the vacuum energy and cosmological “constant”.

### B. Dynamic relaxation of vacuum energy

Let us assume that we can make a sharp kick of the system from its equilibrium state. For quantum liquids (or any other quantum condensed matter) we know the result of the kick: the liquid or superconductor starts to move back to the equilibrium state, and with or without oscillations it finally approaches the equilibrium [54–58]. The same should happen with the quantum vacuum. Let us consider this behavior using the realization of the vacuum  $q$  field in terms of the 4-form field, when  $\mu$  serves as the overall integration constant. We

start with the fully equilibrium vacuum state, which is characterized by the values  $q = q_0$  and  $\mu = \mu_0$  in (4.8). The kick moves the variable  $q$  away from its equilibrium value, while  $\mu$  still remains the same being the overall integration constant,  $\mu = \mu_0$ . In the non-equilibrium state which arises immediately after the kick, the vacuum energy is non-zero and big. If the kick is very sharp, with the time scale of order  $t_P = 1/E_P = \sqrt{G_N}$ , the energy density of the vacuum can reach the Planck-scale value,  $\rho_{\text{vac}} \sim E_P^4$ .

For simplicity we ignore the interaction between the vacuum and matter. Then from the solution of dynamic equations (4.4) and (4.6) with  $\mu = \mu_0$  one finds that after the kick  $q$  does return to its equilibrium value  $q_0$  in the Minkowski vacuum. At late time the relaxation has the following asymptotic behavior: [22]

$$q(t) - q_0 \sim q_0 \frac{\sin \omega t}{\omega t} \quad , \quad \omega t \gg 1 \quad , \quad (5.3)$$

where oscillation frequency  $\omega$  is of the order of the Planck-energy scale  $E_P$ . The gravitational constant  $G$  approaches its Newton value  $G_N$  also with the power-law modulation:

$$G(t) - G_N \sim G_N \frac{\sin \omega t}{\omega t} \quad , \quad \omega t \gg 1 \quad , \quad (5.4)$$

The vacuum energy relaxes to zero in the following way:

$$\rho_{\text{vac}}(t) \propto \frac{\omega^2}{t^2} \sin^2 \omega t \quad , \quad \omega t \gg 1 \quad , \quad (5.5a)$$

For the Planck scale kick, the vacuum energy density after the kick, i.e. at  $t \sim 1/E_P$ , has a Planck-scale value,  $\rho_{\text{vac}} \sim E_P^4$ . According to (5.5a), at present time it must reach the value

$$\bar{\rho}_{\text{vac}}(t_{\text{present}}) \propto \frac{E_P^2}{t_{\text{present}}^2} \sim E_P^2 H^2 \quad , \quad (5.5b)$$

where  $H$  is the Hubble parameter. This value approximately corresponds to the measured value of the cosmological constant.

This, however, can be considered as an illustration of the dynamical reduction of the large value of the cosmological constant, rather than the real scenario of the evolution of the Universe. We did not take into account quantum dissipative effects and the energy exchange between vacuum and matter. Indeed, matter field radiation (matter quanta emission) by the oscillations of the vacuum can be expected to lead to faster relaxation of the initial vacuum energy [59],

$$\rho_{\text{vac}}(t) \propto \Gamma^4 \exp(-\Gamma t) \quad , \quad (5.5c)$$

with a decay rate  $\Gamma \sim \omega \sim E_P$ .

Nevertheless, the cancellation mechanism and example of relaxation provide the following lesson. The Minkowski-type solution appears without fine-tuning of the parameters of the action, precisely because the vacuum is characterized by a constant derivative of the vacuum field rather than by a constant vacuum field itself. As a result, the parameter  $\mu_0$  emerges in (4.8a) as an *integration constant*, i.e., as a parameter of the solution rather than a parameter of the action. Since after the kick the integration constant remains intact, the Universe will return to its equilibrium Minkowski state with  $\rho_{\text{vac}} = 0$ , even if in the non-equilibrium state after the kick the vacuum energy could reach  $\rho_{\text{vac}} \sim E_P^4$ . The idea that the constant derivative of a field may be important for the cosmological constant problem has been suggested earlier by Dolgov [60, 61] and Polyakov [62, 63], where the latter explored the analogy with the Larkin–Pikin effect [64] in solid-state physics.

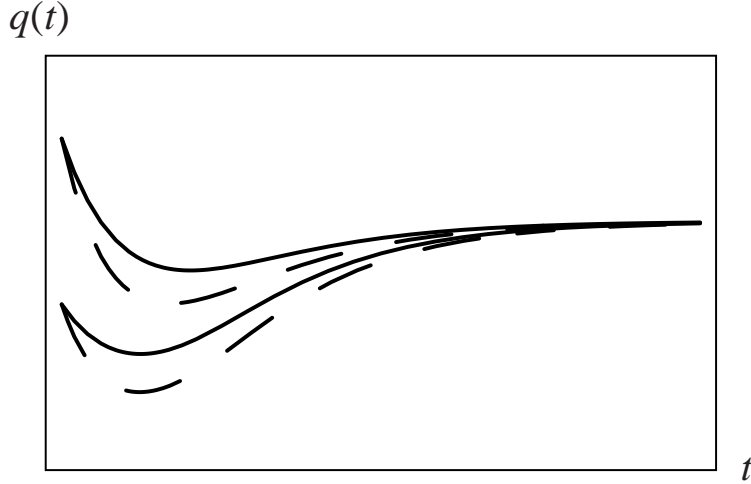


FIG. 4: Aaether-field  $q$  evolution and Minkowski attractor in a spatially flat Friedmann–Robertson–Walker universe in Dolgov model [61] (see [26] for details). The bare cosmological constant is  $\Lambda_{\text{bare}} \sim E_{\text{P}}^4$ . Four numerical solutions correspond to different boundary conditions, but all approach the Minkowski-spacetime solution (5.6). The Minkowski vacuum is an attractor because the vacuum compressibility (3.8) is positive,  $\chi(q_0) > 0$ .

### C. Minkowski vacuum as attractor

The example of relaxation of the vacuum energy in Sec. VB has the principle drawback. Instead of the fine-tuning of the action, which is bypassed in the self-sustained vacuum, we have the fine-tuning of the integration constant. We assumed that originally, before the kick, the Universe was in its Minkowski ground state, and thus the specific value of the integration constant  $\mu = \mu_0$  has been chosen, that fixes the value  $q = q_0$  of the original Minkowski equilibrium vacuum. In the 4-form realization of the vacuum field, any other choice of the integration constant ( $\mu \neq \mu_0$ ) leads asymptotically to a de-Sitter-type solution [22].

To avoid this fine-tuning and obtain the natural relaxation of  $\mu$  to  $\mu_0$ , which as we know occurs in quantum liquids, we must relax the condition on  $\mu$ . It should not serve as an overall integration constant, while remaining the conjugate variable in thermodynamics. Then using the condensed matter experience one may expect that the Minkowski equilibrium vacuum becomes an attractor and the de-Sitter solution with  $\mu \neq \mu_0$  will inevitably relax to Minkowski vacuum with  $\mu = \mu_0$ . This expectation is confirmed in the aether type realization of the vacuum variable in terms of a vector field as discussed in Sec. IIE.

The constant vacuum field  $q$  there appears as the derivative of a vector field in the specific solution  $u_\beta^q$  corresponding to the equilibrium vacuum,  $q g_{\alpha\beta} \equiv \nabla_\alpha u_\beta^q = u_{\alpha\beta}^q$ . In this realization, the effective chemical potential  $\mu \equiv d\epsilon(q)/dq$  plays a role only for the equilibrium states (i.e., for their thermodynamical properties), but  $\mu$  does not appear as an integration constant for the dynamics. Hence, the fine-tuning problem of the integration constant is overcome, simply because there is no integration constant.

The instability of the de-Sitter solution towards the Minkowski one has been already demonstrated by Dolgov [61], who considered the simplest quadratic choices of the Lagrange density of  $u_\beta(x)$ . But his result also holds for the generalized Lagrangian with a generic function  $\epsilon(u_{\alpha\beta})$  in Sec. IIE [26].

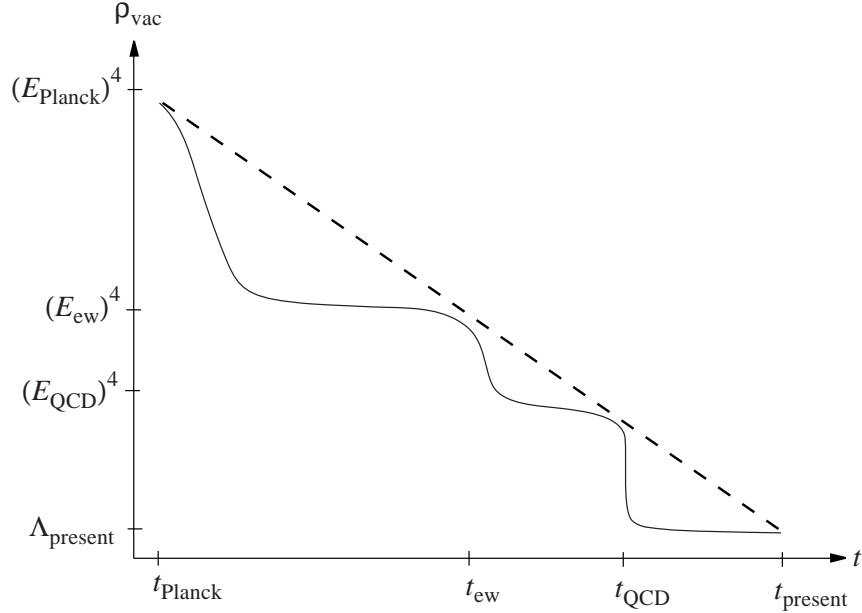


FIG. 5: *Dashed curve*: relaxation according to the relation  $\langle \rho_{\text{vac}}(t) \rangle \sim (E_{\text{Planck}})^2/t^2$  in Eq.(5.5a). *Full curve*: Sketch of the relaxation of the vacuum energy density during the evolution of the Universe according to Ref. [67]. The origin of the current plateau in the vacuum energy  $\Lambda_{\text{present}}$  is discussed in Sec. VD.

The Dolgov scenario does not require the variable gravitational coupling parameter, so that we use  $G(q) = \text{const.}$  In this scenario, for a spatially flat Robertson–Walker metric with cosmic time  $t$  and scale factor  $a(t)$ , the initial de-Sitter-type expansion evolves towards the Minkowski attractor by the following  $t \rightarrow \infty$  asymptotic solution for the aether-type field  $u_\beta = (u_0(t), 0)$ :

$$u_0(t) \rightarrow q_0 t, \quad H(t) \rightarrow 1/t, \quad (5.6)$$

where the Hubble parameter  $H \equiv [da/dt]/a$ . At large cosmic times  $t$ , the curvature terms decay as  $R \sim H^2 \sim 1/t^2$  and the Einstein equations lead to the nullification of the energy-momentum tensor of the  $u_\beta$  field:  $T_{\alpha\beta}[u] = 0$ . Since (5.6) with  $du_0/dt = H u_0$  satisfies the  $q$ -theory *Ansatz*  $u_{\alpha\beta} = q g_{\alpha\beta}$ , the energy-momentum tensor is completely expressed by the single constant  $q$ :  $T_{\alpha\beta}(q) = [\epsilon(q) - q d\epsilon(q)/dq] g_{\alpha\beta}$ . As a result, the equation  $T_{\alpha\beta}(q) = 0$  leads to the equilibrium condition (4.9) for the Minkowski vacuum and to the equilibrium value  $q = q_0$  in (5.6).

Figure 4 shows explicitly the attractor behavior for the simplest case of Dolgov action, with the numerical value of  $q_0$  in (5.6) appearing *dynamically*. This simple version of Dolgov scenario does not appear to give a realistic description of the present Universe [65] and requires an appropriate modification [66]. It nevertheless demonstrates that the compensation of a large initial vacuum energy density can occur dynamically and that Minkowski spacetime can emerge spontaneously, without setting a chemical potential. In other words, an “existence proof” has been given for the conjecture that the appropriate Minkowski value  $q_0$  can result from an attractor-type solution of the field equations. The only condition for the Minkowski vacuum to be an attractor is a positive vacuum compressibility (3.8).

## D. Remnant cosmological constant

Figure 5 demonstrates the possible more realistic scenario with a step-wise relaxation of the vacuum energy density [67]. The vacuum energy density moves from plateau to plateau responding to the possible phase transitions or crossovers in the Standard Model vacuum and follows, on average, the steadily decreasing matter energy density. The origin of the current plateau with a small positive value of the vacuum energy density  $\Lambda_{\text{present}} = \rho_{\text{vac}} \sim (10^{-3} \text{ eV})^4$  is still not clear. It may result from the phenomena, which occur in the infrared. It may come for example from anomalies in the neutrino sector of the quantum vacuum, such as non-equilibrium contribution of the light massive neutrinos to the quantum vacuum [67]; reentrant violation of Lorentz invariance [68] and Fermi point splitting in the neutrino sector [69, 70] (see Sec. VII G). The other possible sources include the QCD anomaly [25, 71–74]; torsion [75]; relaxation effects during the electroweak crossover [23]; etc. Most of these scenarios are determined by the momentum space topology of the quantum vacuum.

## E. Summary and outlook

To study the problems related to quantum vacuum one must search for the proper extension of the current theory of elementary particle physics – the Standard Model. However, many properties of the quantum vacuum can be understood by extending of our experience with self-sustained macroscopic systems to the quantum vacuum. A simple picture of quantum vacuum is based on three assumptions: (i) The quantum vacuum is a self-sustained medium – the system which is stable at zero external pressure, like quantum liquids. (ii) The quantum vacuum is characterized by a conserved charge  $q$ , which is analog of the particle density  $n$  in quantum liquids and which is non-zero in the ground state of the system,  $q = q_0 \neq 0$ . (iii) The quantum vacuum with  $q = q_0$  is a Lorentz-invariant state. This is the only property which distinguishes the quantum vacuum from the condensed-matter quantum liquids.

These assumptions naturally solve the main cosmological constant problem without fine-tuning. In any self-sustained system, relativistic or non-relativistic, in thermodynamic equilibrium at  $T = 0$  the zero-point energy of quantum fields is fully compensated by the microscopic degrees of freedom, so that the relevant energy density is zero in the ground state. This consequence of thermodynamics is automatically fulfilled in any system, which may exist without external environment. This leads to the trivial result for gravity: the cosmological constant in any equilibrium vacuum state is zero. The zero-point energy of the Standard Model fields is automatically compensated by the  $q$ -field that describes the degrees of freedom of the deep quantum vacuum.

These assumptions allow us to suggest that cosmology is the process of equilibration. From the condensed matter experience we know that the ground state of the system serves as an attractor: starting far away from equilibrium, the quantum liquid finally reaches its ground state. The same should occur for the particular case of our Universe: starting far away from equilibrium in a very early phase of universe, the vacuum is moving towards the Minkowski attractor. We are now close to this attractor, simply because our Universe is old. This is a possible reason of the small remnant cosmological constant measured in present time.

The  $q$ -theory transforms the standard cosmological constant problem into the search for the proper decay mechanism of the vacuum energy density and for the proper mechanism of

formation of small remnant cosmological constant. For that we need the theory of dynamics of quantum vacuum. The latter is a new topic in physics waiting for input from theory and observational cosmology. Using several possible realization of the vacuum variable  $q$  we are able to model some features of the vacuum dynamics in a hope that this will allow us to find the generic features and construct the phenomenology of equilibration.

## VI. VACUUM AS TOPOLOGICAL MEDIUM

### A. Topological media

There are two schemes for the classification of states in condensed matter physics and relativistic quantum fields: classification by symmetry and classification by topology.

For the first classification method, a given state of the system is characterized by a symmetry group  $H$  which is a subgroup of the symmetry group  $G$  of the relevant physical laws. The thermodynamic phase transition between equilibrium states is usually marked by a change of the symmetry group  $H$ . This classification reflects the phenomenon of spontaneously broken symmetry. In relativistic quantum fields the chain of successive phase transitions, in which the large symmetry group existing at high energy is reduced at low energy, is in the basis of the Grand Unification models (GUT) [76, 77]. In condensed matter the spontaneous symmetry breaking is a typical phenomenon, and the thermodynamic states are also classified in terms of the subgroup  $H$  of the relevant group  $G$  (see e.g, the classification of superfluid and superconducting states in Refs. [78, 79]). The groups  $G$  and  $H$  are also responsible for classification of topological defects, which are determined by the nontrivial elements of the homotopy groups  $\pi_n(G/H)$  [80].

The second classification method deals with the ground states of the system at zero temperature ( $T = 0$ ). In particle physics it is the classification of quantum vacua. Topological media are systems whose properties are protected by topology and thus are robust to deformations of the action. The universality classes of topological media are determined by momentum-space topology. The latter is also responsible for the type of the effective theory which emerges at low energy. In this sense, topological classification reflects the tendency opposite to GUT, which is called the anti Grand Unification (anti-GUT). In the GUT scheme, the fundamental symmetry of the vacuum state is primary and the phenomenon of spontaneous symmetry breaking gives rise to topological defects. In the anti-GUT scheme the topology is primary, while effective symmetry gradually emerges at low energy [7, 8].

Different aspects of physics of topological matter have been discussed, including topological stability of gap nodes; classification of fully gapped vacua; edge states; Majorana fermions; influence of disorder and interaction; topological quantum phase transitions; intrinsic quantum Hall and spin-Hall effects; quantization of physical parameters; experimental realization; connections with relativistic quantum fields; chiral anomaly; topological Chern-Simons and Wess-Zumino actions; etc.

### B. Gapless topological media

There are two big groups of topological media: with fully gapped fermionic excitations and with gapless fermions.

In 3+1 spacetime, there are four basic universality classes of gapless fermionic vacua protected by topology in momentum space [8, 81]:

(i) Vacua with fermionic excitations characterized by Fermi points (Dirac points, Weyl points, Majorana points, etc.) – points in 3D momentum space at which the energy of fermionic quasiparticle vanishes. Examples are provided by the spin triplet  $p$ -wave superfluid  $^3\text{He-A}$ , Weyl semimetals, and also by the quantum vacuum of Standard Model above the electroweak transition, where all elementary particles are Weyl fermions with Fermi points in the spectrum. This universality class manifests the phenomenon of emergent relativistic quantum fields at low energy: close to the Fermi points the fermionic quasiparticles behave as massless Weyl fermions, while the collective modes of the vacuum interact with these fermions as gauge and gravitational fields.

(ii) Vacua with fermionic excitations characterized by lines in 3D momentum space or points in 2D momentum space. We shall characterize zeroes by their co-dimension – the dimension of  $\mathbf{p}$ -space minus the dimension of the manifold of zeros. Lines in 3D momentum space and points in 2D momentum space have co-dimension 2: since  $3 - 1 = 2 - 0 = 2$ ; compare this with zeroes of class (i) which have co-dimension  $3 - 0 = 3$ . Zeroes of co-dimension 2 are topologically stable only if some special symmetry is obeyed. Examples are provided by the vacuum of the high  $T_c$  cuprate superconductors where the Cooper pairing into a  $d$ -wave state occurs [82] and graphene [16–18, 83]. Nodes in spectrum are stabilized there by the combined effect of momentum-space topology and discrete symmetry.

(iii) Vacua with fermionic excitations characterized by Fermi surfaces. The representatives of this universality class are normal metals and normal liquid  $^3\text{He}$ . This universality class also manifests the phenomenon of emergent physics, though non-relativistic: at low temperature all the metals behave in a similar way, and this behavior is determined by the Landau theory of Fermi liquid – the effective theory based on the existence of Fermi surface. Fermi surface has co-dimension 1: in 3D system it is the surface (co-dimension =  $3 - 2 = 1$ ), in 2D system it is the line (co-dimension =  $2 - 1 = 1$ ), and in 1D system it is the point (co-dimension =  $1 - 0 = 1$ ; in one dimensional system the Landau Fermi-liquid theory does not work, but the Fermi surface survives).

(iv) The Fermi band class, where the energy vanishes in the finite region of the 3D momentum space, and thus zeroes have co-dimension 0. The possible states of this kind has been discussed in [84–86]. In particle physics, the Fermi band or the Fermi ball appears in a 2+1 dimensional nonrelativistic quantum field theory which is dual to a gravitational theory in the anti-de Sitter background with a charged black hole [87]. Topologically stable flat band exists on the surface of the materials with lines of zeroes in bulk [88–90] and in the spectrum of fermion zero modes localized in the core of some vortices [91–93].

### C. Fully gapped topological media

The gapless and gapped vacuum states are interrelated. For example, the quantum phase transition between the fully gapped states with different topology occurs via the intermediate gapless state. The related phenomenon is that the interface between the fully gapped states with different values of topological invariant contains gapless fermions.

The most popular examples of the fully gapped topological matter are topological insulators [9, 10, 94]. The first discussion of the possibility of 3+1 topological insulators can be found in Refs. [95, 96]. The main feature of such materials is that they are insulators in bulk, where electron spectrum has a gap, but there are 2+1 gapless edge states of electrons



on the surface or at the interface between topologically different bulk states as discussed in Ref. [96]. The spin triplet  $p$ -wave superfluid  $^3\text{He-B}$  is another example the fully gapped 3+1 matter with nontrivial topology in momentum space. It has 2+1 gapless quasiparticles living at interfaces between vacua with different values of the topological invariant describing the bulk states of  $^3\text{He-B}$  [97, 98]. The only difference from the topological insulators is that the gapless fermions living on the surface of the topological superfluid and superconductor or at the interface are Majorana fermions. The quantum vacuum of Standard Model below the electroweak transition, i.e. in its massive phase, is the relativistic counterpart of the topological insulators and gapped topological superfluids [99].

Examples of the 2+1 topological fully gapped systems are provided by the films of superfluid  $^3\text{He-A}$  with broken time reversal symmetry [100, 101] and by the planar phase which is time reversal invariant [100, 101]. The topological invariants for 2+1 vacua give rise to quantization of the Hall and spin-Hall conductivity in these films in the absence of external magnetic field (the so-called intrinsic quantum and spin-quantum Hall effects) [100, 102], see Sec. VIII A 3.

#### D. Green's function as an object

For study the topological properties of condensed matter systems, the ideal noninteracting systems are frequently used. Sometimes this is justified, if one can find the effective single-particle Hamiltonian, which emerges at low energy and which reflects the topological properties of the real interacting many-body system. However, in general the primary object for the topological classification of the real systems is the one-electron propagator – Green's function  $G(\omega, \mathbf{p})$ . In principle one can construct the effective Hamiltonian by proper simplification of the Green's function at zero frequency,  $H = G^{-1}(\omega = 0, \mathbf{p})$ . Though in the interacting case the propagator  $G(\mathbf{p}, \omega = 0)$  determines correctly only the zero energy states, see e.g. [103], in some cases it can be used for the construction of the topological invariants alongside with the full Green's function  $G(\omega, \mathbf{p})$ . On the other hand there are situations when the Green's function does not have poles (see [83, 104, 105]). In these cases no well defined energy spectrum exists, and the effective low energy Hamiltonian cannot be introduced. In particle physics, interaction may also lead to the anomalous infrared behavior of propagators. For example, the pole in the Green's function is absent for the so-called unparticles [106, 107]; the phenomenon of quark confinement in QCD can lead to the anomalous infrared behavior of the quark and gluon propagators [108–110]; marginal Green's function of fermions may occur at the black hole horizon [111]; etc. Thus in the interacting systems, all the information on the topology is encoded in the topology of the Green's function matrix, and also in its symmetry. The latter is important, because symmetry supports additional topological invariants, which are absent in the absence of symmetry, see below.

Green's function topology has been used in particular for classification of topologically protected nodes in the quasiparticle energy spectrum of systems of different dimensions including the vacuum of Standard Model in its gapless state [7, 8, 81, 83]; for the classification of the topological ground states in the fully gapped 2 + 1 systems, which experience intrinsic quantum Hall and spin-Hall effects [8, 83, 100, 112–114]; in relativistic quantum field theory of 2 + 1 massive Dirac fermions [115–119] and 3 + 1 massive Dirac fermions [120]; etc. (see also recent papers [121, 122]).

For the topological classification of the gapless vacua, the Green's function is considered

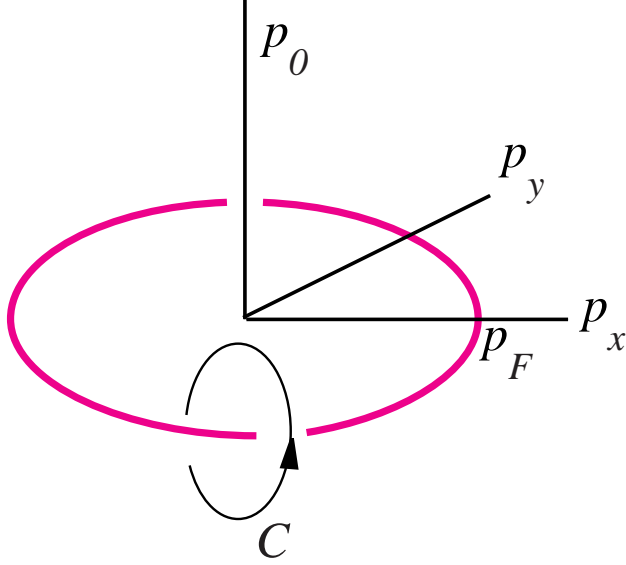


FIG. 6: Fermi surface in 2+1 systems represents the nodes of co-dimension 1. In this case, the Green's function has singularities on line  $p_0 = 0$ ,  $p_x^2 + p_y^2 = p_F^2$  in the three-dimensional space  $(p_0, p_x, p_y)$ . Stability of Fermi surface is protected by the invariant (6.1) which is represented by integral over an arbitrary contour  $C$  around the Green's function singularity. This is applicable to nodes of co-dimension 1 in any  $D+1$  dimension. For  $D = 3$  the nodes form conventional Fermi surface in metals and in normal  $^3\text{He}$ .

at imaginary frequency  $\omega = ip_0$ . This allows us to consider only the relevant singularities in the Green's function and to avoid the singularities on the mass shell, which exist in any vacuum, gapless or fully gapped.

### E. Fermi surface as topological object

Let us start with gapless vacua. The Green's function is generally a matrix with spin indices. In addition, it may have the band indices (in the case of electrons in the periodic potential of crystals). The general analysis [81] demonstrates that topologically stable nodes of co-dimension 1 (Fermi surface in 3+1 metal, Fermi line in 2+1 system or Fermi point in 1+1 system) are described by the group  $Z$  of integers. The winding number  $N_1$ , which is responsible for the topological stability of these node, is expressed analytically in terms of the Green's function [8]:

$$N_1 = \text{tr} \oint_C \frac{dl}{2\pi i} G(p_0, \mathbf{p}) \partial_l G^{-1}(p_0, \mathbf{p}) . \quad (6.1)$$

Here the integral is taken over an arbitrary contour  $C$  around the Green's function singularity in the  $D + 1$  momentum-frequency space. See Fig. 6 for  $D=2$ . Example of the Green's function in any dimension  $D$  is scalar function  $G^{-1}(\omega, \mathbf{p}) = ip_0 - v_F(|\mathbf{p}| - p_F)$ . For  $D = 2$ , the singularity with winding number  $N_1 = 1$  is on the line  $p_0 = 0$ ,  $p_x^2 + p_y^2 = p_F^2$ , which represents the one-dimensional Fermi surface.

Due to nontrivial topological invariant, Fermi surface survives the perturbative interaction and exists even in marginal and Luttinger liquids without poles in the Green's function, where quasiparticles are not well defined.

## VII. VACUUM IN A SEMI-METAL STATE

For our Universe, which obeys the Lorentz invariance, only those vacua are important that are either Lorentz invariant, or acquire the Lorentz invariance as an effective symmetry emerging at low energy. This excludes the vacua with Fermi surface and Fermi lines and leaves the class of vacua with Fermi point of chiral type, in which fermionic excitations behave as left-handed or right-handed Weyl fermions [7, 8], and the class of vacua with the nodal point obeying  $Z_2$  topology, where fermionic excitations behave as massless Majorana neutrinos [81, 83].

### A. Fermi points in 3+1 vacua

For relativistic quantum vacuum of our 3+1 Universe the Green's function singularity of co-dimension 3 is relevant. They are described by the following topological invariant expressed via integer valued integral over the surface  $\sigma$  around the singular point in the 4-momentum space  $p_\mu = (p_0, \mathbf{p})$ [8]:

$$N_3 = \frac{e_{\alpha\beta\mu\nu}}{24\pi^2} \text{tr} \int_{\sigma} dS^\alpha G \partial_{p_\beta} G^{-1} G \partial_{p_\mu} G^{-1} G \partial_{p_\nu} G^{-1}. \quad (7.1)$$

If the invariant is nonzero, the Green's function has point singularity inside the surface  $\sigma$  – the Fermi point. If the topological charge is  $N_3 = +1$  or  $N_3 = -1$ , the Fermi point represents the so-called conical Dirac point, but actually describes the chiral Weyl fermions. This is the consequence of the so-called Atiyah-Bott-Shapiro construction [81], which leads to the following general form of expansion of the inverse fermionic propagator near the Fermi point with  $N_3 = +1$  or  $N_3 = -1$ :

$$G^{-1}(p_\mu) = e_\alpha^\beta \Gamma^\alpha (p_\beta - p_\beta^{(0)}) + \dots. \quad (7.2)$$

Here  $\Gamma^\mu = (1, \sigma_x, \sigma_y, \sigma_z)$  are Pauli matrices (or Dirac matrices in the more general case); the expansion parameters are the vector  $p_\beta^{(0)}$  indicating the position of the Fermi point in momentum space where the Green's function has a singularity, and the matrix  $e_\alpha^\beta$ ; ellipsis denote higher order terms in expansion.

### B. Emergent fermionic matter

The equation (7.2) can be continuously deformed to the simple one, which describes the relativistic Weyl fermions

$$G^{-1}(p_\mu) = ip_0 + N_3 \boldsymbol{\sigma} \cdot \mathbf{p} + \dots, \quad (7.3)$$

where the position of the Fermi point is shifted to  $p_\beta^{(0)} = 0$  and ellipsis denote higher order terms in  $p_0$  and  $\mathbf{p}$ ; the matrix  $e_\alpha^\beta$  is deformed to unit matrix. This means that close to

the Fermi point with  $N_3 = +1$ , the low energy fermions behave as right handed relativistic particles, while the Fermi point with  $N_3 = -1$  gives rise to the left handed particles.

The equation (7.3) suggests the effective Weyl Hamiltonian

$$H_{\text{eff}} = N_3 \boldsymbol{\sigma} \cdot \mathbf{p}. \quad (7.4)$$

However, the infrared divergences may violate the simple pole structure of the propagator in Eq.(7.3). In this case in the vicinity of Fermi point one has

$$G(p_\mu) \propto \frac{-ip_0 + N_3 \boldsymbol{\sigma} \cdot \mathbf{p}}{(p^2 + p_0^2)^\gamma}, \quad (7.5)$$

with  $\gamma \neq 1$ . This modification does not change the topology of the propagator: the topological charge of singularity is  $N_3$  for arbitrary parameter  $\gamma$  [83]. For fermionic unparticles one has  $\gamma = 5/2 - d_U$ , where  $d_U$  is the scale dimension of the quantum field [106, 107].

For  $N_3 = \pm 2$ , the spectrum of (quasi)particles in the vicinity of singularity depends on symmetry. One may obtain either two Weyl fermions or exotic massless fermions with non-linear dispersion at low energy: semi-Dirac fermions with linear dispersion in one direction and quadratic dispersion in the two others [8, 68, 83].

$$E(\mathbf{p}) \approx \pm \sqrt{c^2 p_z^2 + \left(\frac{p_\perp^2}{2m}\right)^2}. \quad (7.6)$$

Similar consideration for the 2+1 systems may lead to semi-Dirac fermions and to fermions with quadratic dispersion at low energy [83, 123, 124]

$$E(\mathbf{p}) \approx \pm \frac{p^2}{2m}. \quad (7.7)$$

For the higher values of topological charge, the spectrum becomes even more interesting (see e.g. Refs. [125, 126] for 2+1 systems). But if the relativistic invariance is obeyed, or under the special discrete symmetry, the nonzero invariant  $N_3$  corresponds to  $N_3$  species of Weyl fermions near the Fermi point.

The main property of the vacua with Dirac points is that according to (7.3), close to the Fermi points the massless relativistic fermions emerge. This is consistent with the fermionic content of our Universe, where all the elementary particles – left-handed and right-handed quarks and leptons – are Weyl fermions. Such a coincidence demonstrates that the vacuum of Standard Model in its massless phase is the topological medium of the Fermi point universality class. This solves the hierarchy problem, since the value of the masses of elementary particles in the vacua of this universality class is strictly zero.

Let us suppose for a moment, that there is no topological invariant which protects massless fermions. Then the Universe is fully gapped and the natural masses of fermions must be on the order of Planck energy scale:  $M \sim E_P \sim 10^{19}$  GeV. In such a natural Universe, where all masses are of order  $E_P$ , all fermionic degrees of freedom are completely frozen out because of the Boltzmann factor  $e^{-M/T}$ , which is about  $e^{-10^{16}}$  at the temperature corresponding to the highest energy reached in accelerators. There is no fermionic matter in such a Universe at low energy. That we survive in our Universe is not the result of the anthropic principle (the latter chooses the Universes which are fine-tuned for life but have an extremely low probability). Our Universe is also natural and its vacuum is generic, but it belongs to a different universality class of vacua – the vacua with Fermi points. In such vacua the masslessness of fermions is protected by topology (combined with symmetry, see below).

### C. Emergent gauge fields

The vacua with Fermi-point suggest a particular mechanism for emergent symmetry. The Lorentz symmetry is simply the result of the linear expansion: this symmetry becomes better and better when the Fermi point is approached and the non-relativistic higher order terms in Eq.(7.3) may be neglected. This expansion demonstrates the emergence of the relativistic spin, which is described by the Pauli matrices. It also demonstrates how gauge fields and gravity emerge together with chiral fermions. The expansion parameters  $p_\beta^{(0)}$  and  $e_\alpha^\beta$  may depend on the space and time coordinates and they actually represent collective dynamic bosonic fields in the vacuum with Fermi point. The vector field  $p_\beta^{(0)}$  in the expansion plays the role of the effective  $U(1)$  gauge field  $A_\beta$  acting on fermions.

For the Fermi points with topological charge  $N_3 > 1$  the situation depends on the symmetry of the system. In the case, when the spectrum corresponds to several species of relativistic Weyl fermions, the shift  $p_\beta^{(0)}$  becomes the matrix field; it gives rise to effective non-Abelian (Yang-Mills)  $SU(N_3)$  gauge fields emerging in the vicinity of Fermi point, i.e. at low energy [8]. For example, the Fermi point with  $N_3 = 2$  may give rise to the effective  $SU(2)$  gauge field in addition to the effective  $U(1)$  gauge field

$$G^{-1}(p_\mu) = e_\alpha^\beta \Gamma^\alpha (p_\beta - g_1 A_\beta - g_2 \mathbf{A}_\beta \cdot \boldsymbol{\tau}) + \text{higher order terms}, \quad (7.8)$$

where  $\boldsymbol{\tau}$  are Pauli matrices corresponding to the emergent isotopic spin. This is what happens in superfluid  $^3\text{He-A}$ . In the case, when the symmetry leads to exotic fermions with the non-linear spectrum  $E \sim \pm p^{N_3}$ , the quantum electrodynamics with anisotropic scaling emerges [127, 128], which is similar to the quantum gravity with anisotropic scaling suggested by Hořava [129–131].

### D. Emergent gravity

The matrix field  $e_\alpha^\beta$  in (7.8) acts on the (quasi)particles as the field of vierbein, and thus describes the emergent dynamical gravity field. As a result, close to the Fermi point, matter fields (all ingredients of Standard Model: chiral fermions and Abelian and non-Abelian gauge fields) emerge together with geometry, relativistic spin, Dirac matrices, and physical laws: Lorentz and gauge invariance, equivalence principle, etc. In such vacua, gravity emerges together with matter. If this Fermi point mechanism of emergence of physical laws works for our Universe, then the so-called “quantum gravity” does not exist. The gravitational degrees of freedom can be separated from all other degrees of freedom of quantum vacuum only at low energy.

In this scenario, classical gravity is a natural macroscopic phenomenon emerging in the low-energy corner of the microscopic quantum vacuum, i.e. it is a typical and actually inevitable consequence of the coarse graining procedure. It is possible to quantize gravitational waves to obtain their quanta – gravitons, since in the low energy corner the results of microscopic and effective theories coincide. It is also possible to obtain some (but not all) quantum corrections to Einstein equation and to extend classical gravity to the semiclassical level. But one cannot obtain “quantum gravity” by quantization of Einstein equations, since all other degrees of freedom of quantum vacuum will be missed in this procedure.

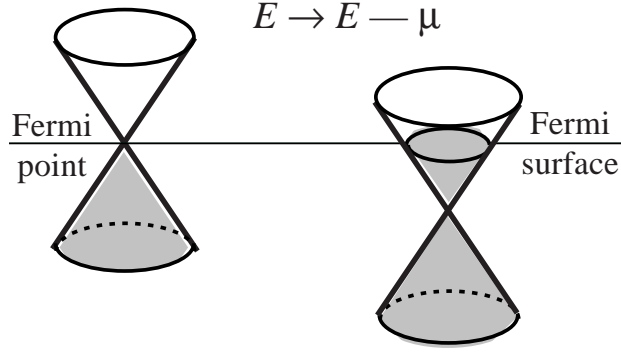


FIG. 7: Fermi surface is formed from the Fermi point at finite chemical potential of chiral fermions, when the Fermi point is moved away from the zero energy level.

### E. Topological invariant for specific Fermi surface

If the symmetry which fixes the position of conical (Dirac) point at zero energy level is violated, the conical point moves from the zero energy position upward or downward from the chemical potential and the Fermi surface is formed. This is shown in Fig. 7. This Fermi surface has specific property: in addition to the local charge  $N_1$  in (6.1), which characterizes singularities at the Fermi surface, it is described by the global charge  $N_3$  in (7.1). The integral in (7.1) is now over the surface  $\sigma$  which embrace whole Fermi sphere. The Fermi surface with the global topological charge appears in superfluid  $^3\text{He-A}$  in the presence of mass flow [8]; it is also discussed for the 2+1 systems in relation to the gapless states on the surface of 3+1 insulators [132].

The “collision” of the Fermi surfaces in momentum space leads to the redistribution of the global topological charges  $N_3$  between the Fermi surfaces when they touch each other,  $(+1) + (-1) \rightarrow 0 + 0$  [69, 83]. Such collision, at which the Fermi surface loses its global charge  $N_3$ , represents the topological quantum phase transition (see Fig. 8), one of numerous types of transitions induced by topology in momentum space [83].

### F. Topological invariant protected by symmetry in semi-metal state

We assume that Standard Model contains equal number of right and left Weyl fermions,  $n_R = n_L = 8n_g$ , where  $n_g$  is the number of generations (we do not consider Standard Model with Majorana fermions, and assume that in the insulating state of Standard Model neutrinos are Dirac fermions). For such Standard Model the topological charge in (7.1) vanishes,  $N_3 = 8n_g - 8n_g = 0$ . Thus the masslessness of the Weyl fermions is not protected by the invariant (7.1), and arbitrary weak interaction may result in massive particles.

However, there is another topological invariant, which takes into account the symmetry of the vacuum. The gapless state of the vacuum with  $N_3 = 0$  can be protected by the following integral [8]:

$$N_3^K = \frac{e_{\alpha\beta\mu\nu}}{24\pi^2} \text{tr} \left[ K \int_{\sigma} dS^{\alpha} G \partial_{p_{\beta}} G^{-1} G \partial_{p_{\mu}} G^{-1} G \partial_{p_{\nu}} G^{-1} \right]. \quad (7.9)$$

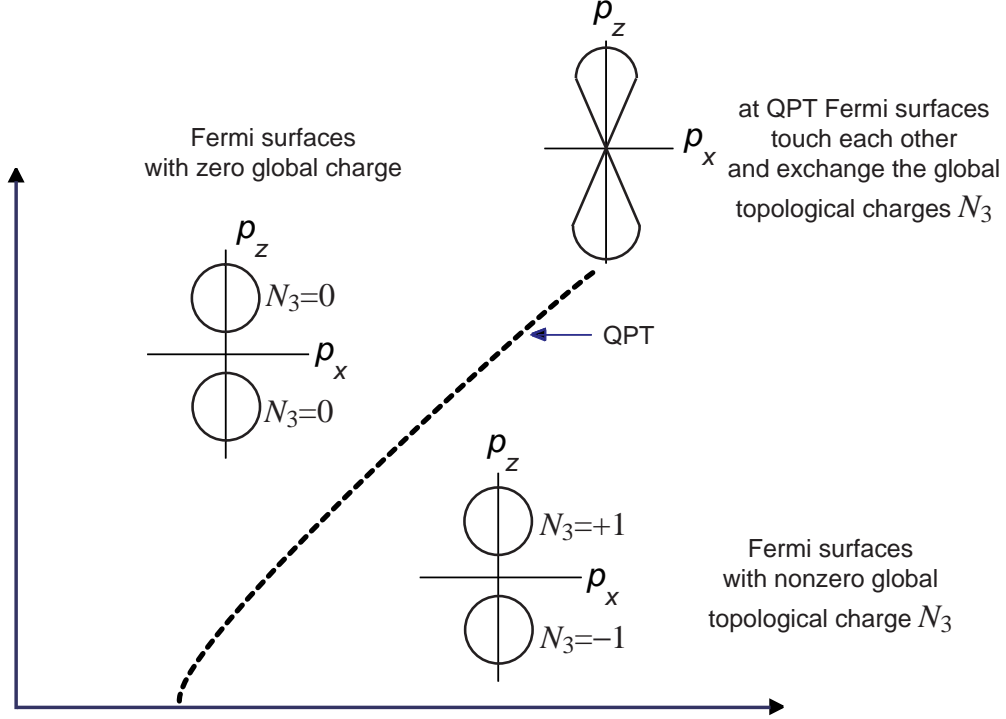


FIG. 8: Dashed line represents the topological quantum phase transition in the model [69]. The vacua on both sides of the transition have Fermi surfaces. On the right side of transition, Fermi surfaces have nonzero global topological charges  $N_3 = +1$  and  $N_3 = -1$ . At the transition, the Fermi surfaces “collide”, and their topological global charges  $N_3$  annihilate each other. On the left side of transition, Fermi surfaces become globally trivial,  $N_3 = 0$ , but retain their local topological charge  $N_1$  in (6.1).

where  $K_{ij}$  is the matrix of some symmetry transformation, which either commutes or anticommutes with the Green’s function matrix. In Standard Model there are two relevant symmetries, both are the  $Z_2$  groups,  $K^2 = 1$ . One of them is the center subgroup of  $SU(2)_L$  gauge group of weak rotations of left fermions, where the element  $K$  is the gauge rotation by angle  $2\pi$ ,  $K = e^{i\pi\tilde{\tau}_{3L}}$ . The other one is the group of the hypercharge rotation by angle  $6\pi$ ,  $K = e^{i6\pi Y}$ . In the  $G(224)$  Pati-Salam extension of the  $G(213)$  group of Standard Model, this symmetry comes as combination of the  $Z_2$  center group of the  $SU(2)_R$  gauge group for right fermions,  $e^{i\pi\tilde{\tau}_{3R}}$ , and the element  $e^{3\pi i(B-L)}$  of the  $Z_4$  center group of the  $SU(4)$  color group – the  $P_M$  parity (on the importance of the discrete groups in particle physics see [133, 134] and references therein). Each of these two  $Z_2$  symmetry operations changes sign of left spinor, but does not influence the right particles. Thus these matrices are diagonal,  $K_{ij} = \text{diag}(1, 1, \dots, -1, -1, \dots)$ , with eigen values 1 for right fermions and  $-1$  for left fermions.

In the symmetric phase of Standard Model, both matrices commute with the Green’s function matrix  $G_{ij}$ , as a result  $N_3^K$  is topological invariant: it is robust to deformations of Green’s function which preserve the symmetry  $K$ . The value of this invariant  $N_3^K = 16n_g$ , which means that all  $16n_g$  fermions are massless if the symmetry  $K$  is obeyed.

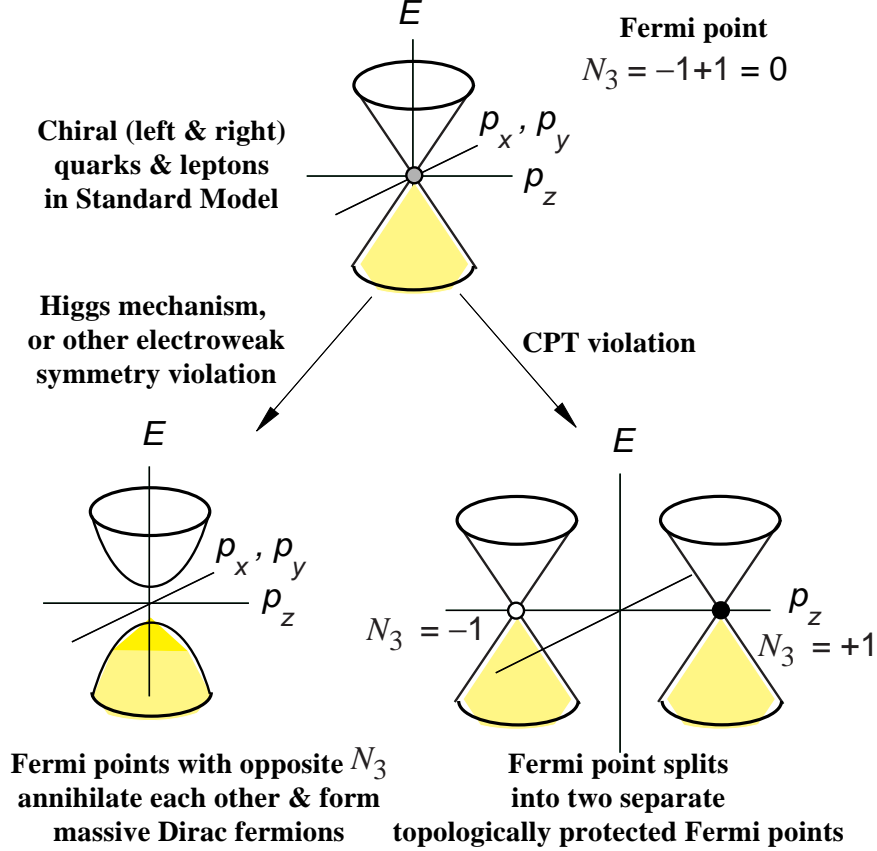


FIG. 9: (*top*): In Standard Model the Fermi points with positive  $N_3 = +1$  and negative  $N_3 = -1$  topological charges are at the same point  $\mathbf{p} = 0$ , forming the marginal Fermi point with  $N_3 = 0$ . Symmetry  $K$  between the Fermi points prevents their mutual annihilation giving rise to the topological invariant (7.9) with  $N_3^K = 16n_g$ . Figure illustrates the simple case with two Weyl fermions, one with  $N_3 = +1$  and another with  $N_s = -1$ , when the invariant  $N_3^K = 2$  protects the marginal Fermi point with  $N_3 = +1 - 1 = 0$ . (*bottom left*): If symmetry  $K$  is violated or spontaneously broken, Weyl points annihilate each other and Dirac mass is formed. (*bottom right*): If Lorentz invariance is violated or spontaneously broken, the marginal Fermi point splits [135]. The topological quantum phase transition between the state with Dirac mass and the state with splitted Dirac points have been observed in cold Fermi gas [136].

### G. Higgs mechanism vs splitting of Fermi points

The gapless vacuum of Standard Model is supported by combined action of topology and symmetry  $K$ , and also by the CPT and Lorentz invariance which keep all the Fermi points at  $\mathbf{p} = 0$ .

Explicit violation or spontaneous breaking of one of these symmetries transforms the vacuum of the Standard Model into one of the two possible vacua. If, for example, the  $K$  symmetry is broken, the invariant (7.9) supported by this symmetry ceases to exist, and the Fermi point disappears. All  $16n_g$  fermions become massive (Fig. 9 *bottom left*). This is assumed to happen below the symmetry breaking electroweak transition caused by Higgs mechanism where quarks and charged leptons acquire the Dirac masses.



If, on the other hand, the CPT symmetry is violated, the marginal Fermi point splits into topologically stable Fermi points with non-zero invariant  $N_3$ , which protects massless chiral fermions (Fig. 9 *bottom right*). Since the invariant  $N_3$  does not depend on symmetry, the further symmetry breaking cannot destroy the nodes. One can speculate that in the Standard Model the latter may happen with the electrically neutral leptons, the neutrinos [69]. Fermi point splitting in the neutrino sector may serve as an example of spontaneous breaking of Lorentz symmetry [70, 137]. It may also provide a new source of T and CP violation in the leptonic sector, which may be relevant for the creation of the observed cosmic matter-antimatter asymmetry [138]. Examples of splitting of Fermi and Majorana points in condensed matter are discussed in the review paper [83].

## H. Fermi points in condensed matter

### 1. Chiral superfluid $^3\text{He-A}$

The discovery of superfluid  $^3\text{He}$  in 1972 [79, 139] marked the first condensed matter realization of topological medium. Both phases of superfluid  $^3\text{He}$  (gapless  $^3\text{He-A}$  and fully gapped  $^3\text{He-B}$ ) are topological superfluids. The chiral superfluid  $^3\text{He-A}$  with broken time reversal symmetry has the following simplified Green's function for each spin projection

$$G^{-1}(\omega, \mathbf{p}) = ip_0 + \tau_3 \left( \frac{p^2}{2m} - \mu \right) + c(\tau_1 p_x + \tau_2 p_y), \quad (7.10)$$

where  $\tau_i$  are Pauli matrices of Bogolyubov-Nambu spin. For  $\mu > 0$  there are two Weyl points at  $p_x = p_y = 0$  and  $p_z = \pm\sqrt{2m\mu}$  with  $N_3 = \pm 1$ . For  $\mu < 0$  the vacuum is fully gapped. Thus at  $\mu = 0$  there is a topological quantum phase transition from the gapless to gapped vacuum [101]. At the transition, i.e. at  $\mu = 0$ , there is a marginal (topologically trivial) Fermi point with  $N_3 = 0$ , situated at  $\mathbf{p} = 0$ , just as in Fig. 9 *top*. At  $\mu > 0$ , this marginal Fermi point splits in two topologically protected Weyl points with  $N_3 = \pm 1$ , Fig. 9 *bottom right*.

### 2. Cube of Fermi points

The vacuum of the Standard Model contains 16 Weyl points in each generation. Example of the condensed matter system with Weyl Dirac points is provided by the  $\alpha$ -phase of spin-triplet  $p$ -wave superfluid [79] and spin-singlet  $d$ -wave superconductor [78]. The latter has the following simplified Green's function:

$$G^{-1}(\omega, \mathbf{p}) = ip_0 + \tau_3 \left( \frac{p^2}{2m} - \mu \right) + \tau_1(2p_z^2 - p_y^2 - p_x^2) + \tau_2\sqrt{3}(p_y^2 - p_x^2). \quad (7.11)$$

The spectrum has 8 point nodes – Weyl points with  $N_3 = +1$  and  $N_3 = -1$  situated on vertices of cube in momentum space in Fig. 10 [78]. At  $\mu = 0$  all eight Weyl points collapse to the marginal Fermi point with  $N_3 = 0$  situated at  $\mathbf{p} = 0$ .

The  $\alpha$ -phase is the analog of the 3+1 “graphene” in relativistic quantum fields [140].

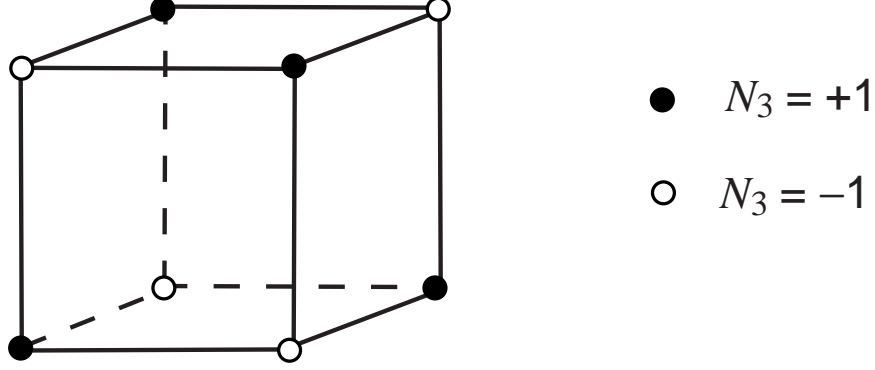


FIG. 10: Cube of Fermi points (Weyl points) in momentum space in the  $\alpha$ -phase of spin-triplet  $p$ -wave superfluid [79] and spin-singlet  $d$ -wave superconductor [78]. Filled circles denote Weyl points with positive topological charge  $N_3 = +1$  and thus with right-handed fermions living in the vicinity of these Weyl points. Open circles denote the nodes with negative topological charge  $N_3 = -1$  and the left-handed fermions. This is analogous of the four-dimensional graphene-like vacuum in a model of relativistic quantum field theory, which is characterized by Dirac points on vertices of the 4D hypercube [140].

### 3. Time reversal invariant planar phase

The planar phase of spin-triplet superfluid/superconductor is characterized by the following simplified  $4 \times 4$  Green's function matrix

$$G^{-1}(\omega, \mathbf{p}) = ip_0 + \tau_3 \left( \frac{p^2}{2m} - \mu \right) + \tau_1(\sigma_x p_x + \sigma_y p_y). \quad (7.12)$$

As distinct from the chiral  $^3\text{He-A}$  and  $\alpha$ -phase, the planar phase obeys time-reversal invariance. Two nodes in spectrum, at  $p_x = p_y = 0$  and  $p_z = \pm\sqrt{2m\mu}$ , have zero topological charges (7.1),  $N_3 = 0$ . But these nodes are protected by the topological charge  $N_3^K = \pm 2$  in (7.9), where the corresponding symmetry of the planar phase is  $K = \tau_3 \sigma_z$ . This matrix  $K$  commutes with the Green's function matrix.

### 4. Gapless 2+1 vacua

In addition to the Fermi surface class of singularities of co-dimension 1, in 2+1 systems there is a class of vacua with singularities of co-dimension 2: points in 2D momentum space. They corresponds to lines in 3+1 vacua, which also have co-dimension 2. According to [81]: if no symmetry is imposed there is no singularity in Green's function, which is topologically stable. For real fermions, the  $Z_2$  singularities of co-dimension 2 are possible [81]. This means that two such singularities may collapse forming the fully gapped state,  $1 + 1 = 0$ . In 2+1 dimension, the fermions near such singularity behave as Majorana fermions. Some symmetries allow to have the Fermi points of co-dimension 2 with group  $Z$ . The corresponding invariant protected by symmetry  $K$  [83, 141, 142] is:

$$N_2^K = \frac{1}{4\pi i} \text{tr} \oint_C dl K G(\omega = 0, \mathbf{p}) \partial_l G^{-1}(\omega = 0, \mathbf{p}), \quad (7.13)$$

where  $C$  is contour around the Fermi point in 2D momentum space  $(p_x, p_y)$ , or around the Fermi surface if the Fermi point expands to the Fermi surface. Examples are graphene and  $d$ -wave cuprate superconductor. For the latter the simplified Green's function has the form

$$G^{-1}(\omega, p_x, p_y) = ip_0 + \tau_3 \left( \frac{p_x^2 + p_y^2}{2m} - \mu \right) + \tau_1(p_y^2 - p_x^2), \quad (7.14)$$

with  $K = \tau_2$ , which anti-commutes with Green's function at zero frequency. The  $d$ -wave superconductor has 4 point nodes at  $|p_x| = |p_y| = (m\mu)^{1/2}$  with  $N_2^K = \pm 1$ . The nodes do not disappear under deformation which preserves symmetry  $K$ . For example, the deformation which violates the 4-fold symmetry of (7.14)

$$G^{-1}(\omega, p_x, p_y) = ip_0 + \tau_3 \left( \frac{p_x^2 + p_y^2}{2m} - \mu \right) + \tau_1(p_y^2 - ap_x^2), \quad (7.15)$$

does not destroy nodes, but shifts positions of nodes. The nodes disappear only at large deformation, when the deformation parameter  $a$  in (7.15) crosses zero, and the topological quantum phase transition occurs. At  $a = 0$  nodes collapse forming two marginal nodes with  $N_2^K = 0$  at  $p_y = 0$ ,  $p_x = \pm(2m\mu)^{1/2}$ , and at  $a < 0$  the fully gapped state is formed [83].

Note that the inverse propagator at  $p_0 = 0$  has all the properties of a free-fermion Hamiltonian, whose topology was discussed in Ref. [143]. But this is actually the effective Hamiltonian, which emerges in the original interacting system (see [103, 144]).

Another class of  $2 + 1$  Fermi points arise at the boundary between the  $3 + 1$  gapped systems with different topological charges. Such points are described by the difference of bulk invariants [144]. This is analogous to the index theorem for fermion zero modes on strings [145] and vortices [8].

## VIII. VACUUM IN STATE OF TOPOLOGICAL INSULATOR

The special role in classification of topological systems is played by dimensional reduction. The dimensional reduction allows us to use for classification of the gapped systems the scheme, which was suggested by Hořava for the classification of the topologically nontrivial nodes in spectrum [81]. The fully gapped vacua in  $D + 1$  space-time are described by the same invariants as nodes of co-dimension  $D + 1$  [8]. For example, the winding number  $N_1$  in (6.1) which describes zeroes of co-dimension 1 (conventional Fermi surface in  $D = 3$  momentum space), also describes the  $D = 0$  gapped systems. The integral (6.1) is now over imaginary frequency:

$$\tilde{N}_1 = \text{tr} \int \frac{dp_0}{2\pi i} G(p_0) \partial_{p_0} G^{-1}(p_0). \quad (8.1)$$

This integer-valued index now shows the difference between the numbers of the positive and negative energy levels of zero-dimensional system.

The classification must be supplemented by the symmetry consideration, which leads to the additional invariants of the type

$$\tilde{N}_1^K = \text{tr} \int \frac{dp_0}{2\pi i} K G(p_0) \partial_{p_0} G^{-1}(p_0), \quad (8.2)$$

where  $K$  is the symmetry operator, which commutes or anti-commutes with the Green's function. The emergent symmetries might also appear within some topological classes.

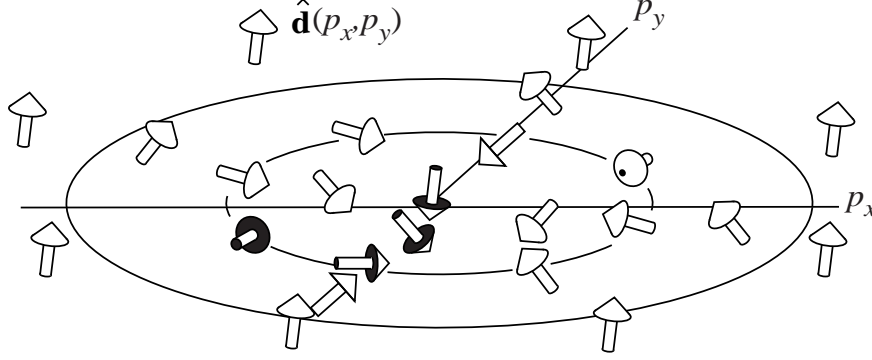


FIG. 11: Skyrmion in  $\mathbf{p}$ -space with momentum space topological charge  $\tilde{N}_3 = -1$  in (8.5). It describes topologically non-trivial fully gapped vacua in 2+1 systems, which have non-singular Green's function. Vacua with nonzero  $\tilde{N}_3$  have topologically protected gapless edge states. The nonzero topological charge leads also to quantization of Hall and spin Hall conductance.

### A. 2+1 fully gapped vacua

#### 1. $^3\text{He-A}$ film: 2+1 chiral superfluid

Let us start with  $D = 2$ . The fully gapped ground states (vacua) in 2+1 or quasi 2+1 thin films of  $^3\text{He-A}$  are characterized by the invariant obtained by dimensional reduction from the topological invariant describing the nodes of co-dimension 3. This is the invariant  $N_3$  for the Fermi point in (7.1), which is now over the (2+1)-dimensional momentum-frequency space  $(p_x, p_y, p_0)$ :

$$\tilde{N}_3 = \frac{e_{ijk}}{24\pi^2} \text{tr} \left[ \int d^2p dp_0 G \partial_{p_i} G^{-1} G \partial_{p_j} G^{-1} G \partial_{p_k} G^{-1} \right]. \quad (8.3)$$

This equation (8.3) was first introduced in relativistic 2 + 1 theories [115–117] and then independently for the film of  $^3\text{He-A}$  in condensed matter [100, 150], where it was inspired by the dimensional reduction from the Fermi point, see [101]. The topological invariant for the general case of the insulating relativistic vacua in even space dimension  $D = 2n$  has been considered in [146–148]. In simple case of the  $2 \times 2$  matrix, the Green's function can be expressed in terms of the three-dimensional vector  $\mathbf{d}(p_x, p_y)$ ,

$$G^{-1}(\omega, p_x, p_y) = ip_0 + \boldsymbol{\tau} \cdot \mathbf{d}(p_x, p_y), \quad (8.4)$$

Example of the  $\mathbf{d}$ -vector configuration, which corresponds to the topologically nontrivial vacuum is presented in Fig. 11. This is the momentum-space analog of the topological object in real space – skyrmion (skyrmions in real space are described by the relative homotopy groups [149]). The winding number of the momentum-space skyrmion is [150]

$$\tilde{N}_3 = \frac{1}{4\pi} \int d^2p \, \hat{\mathbf{d}} \cdot \left( \frac{\partial \hat{\mathbf{d}}}{\partial p_x} \times \frac{\partial \hat{\mathbf{d}}}{\partial p_y} \right), \quad (8.5)$$

where  $\hat{\mathbf{d}} = \mathbf{d}/|\mathbf{d}|$  is unit vector. For a single layer of the  $^3\text{He-A}$  film and for one spin

projection, the simplified Green's function has the form:

$$G^{-1}(\omega, \mathbf{p}) = i p_0 + \boldsymbol{\tau} \cdot \mathbf{d}(\mathbf{p}) = i p_0 + \tau_3 \left( \frac{p_x^2 + p_y^2}{2m} - \mu \right) + \tau_1 p_x + \tau_2 p_y, \quad (8.6)$$

For  $\mu > 0$  the topological charge  $\tilde{N}_3 = 1$  and for  $\mu < 0$  the topological charge is  $\tilde{N}_3 = 0$ . That is why at  $\mu = 0$  there is a topological quantum phase transition between the topological superfluid at  $\mu > 0$  and non-topological superfluid at  $\mu < 0$  [101].

In general case of multilayered  $^3\text{He-A}$ , topological charge  $\tilde{N}_3$  may take any integer value of group  $Z$ . This charge determines quantization of Hall and spin-Hall conductance (see Sec. VIII A 3), and the quantum statistics of the topological objects – real-space skyrmions [100–102, 150]. For  $\tilde{N}_3 = 4k + 1$  and  $\tilde{N}_3 = 4k + 3$ , skyrmion is anyon; for  $\tilde{N}_3 = 4k + 2$  it is fermion; and for  $\tilde{N}_3 = 4k$  it is boson [101]. This demonstrates the importance of the  $Z_2$  and  $Z_4$  subgroups of the group  $Z$  in classification of topological matter; and also provides an example of the interplay of momentum-space and real-space topologies. Applications of topology in combined  $(\mathbf{p}, \mathbf{r})$ -space see in [8, 97, 121, 151–155], in particular it is responsible for the topologically protected spectrum of fermions living on topological objects such as walls, strings and monopoles.

## 2. Planar phase: time reversal invariant gapped vacuum

In case when some symmetry is present, additional invariants appear, which correspond to dimensional reduction of invariant  $N_3^K$  in (7.9):

$$\tilde{N}_3^K = \frac{e_{ijk}}{24\pi^2} \text{tr} \left[ \int d^2 p d p_0 K G \partial_{p_i} G^{-1} G \partial_{p_j} G^{-1} G \partial_{p_k} G^{-1} \right], \quad (8.7)$$

where as before, the matrix  $K$  commutes or anticommutes with the Green's function matrix. Example of the symmetric 2 + 1 gapped state with  $\tilde{N}_3^K$  is the film of the planar phase of superfluid  $^3\text{He}$  [100, 101]. In the single layer case, the simplest expression for the Green's function is

$$G^{-1}(\omega, p_x, p_y) = i p_0 + \tau_3 \left( \frac{p_x^2 + p_y^2}{2m} - \mu \right) + \tau_1 (\sigma_x p_x + \sigma_y p_y), \quad (8.8)$$

with  $K = \tau_3 \sigma_z$  commuting with the Green's function. This state is time reversal invariant. It has  $\tilde{N}_3 = 0$  and  $\tilde{N}_3^K = 2$ . For the general case of the quasi 2D film with multiple layers of the planar phase, the invariant  $\tilde{N}_3^K$  belongs to the group  $Z$ , i.e.  $\tilde{N}_3^K = 2k$ .

## 3. Quantum spin Hall effect

The topological invariants  $\tilde{N}_3$  and  $\tilde{N}_3^K$  give rise to quantization of Hall and spin-Hall conductance in 2+1 gapped systems. There are several types of responses of spin current and electric current to transverse forces which are quantized in 2+1 systems under appropriate conditions. The most familiar is the conventional quantum Hall effect (QHE) [156]. It is quantized response of the particle current to the transverse force, say to transverse gradient of chemical potential,  $\mathbf{J} = \sigma_{xy} \hat{\mathbf{z}} \times \nabla \mu$ . In the electrically charged systems this is the quantized response of the electric current  $\mathbf{J}^e$  to transverse electric field  $\mathbf{J}^e = e^2 \sigma_{xy} \hat{\mathbf{z}} \times \mathbf{E}$ .

The other effects involve the spin degrees of freedom. An example is the mixed spin quantum Hall effect: quantized response of the particle current  $\mathbf{J}$  (or electric current  $\mathbf{J}^e$ ) to transverse gradient of magnetic field  $\mathbf{H}$  interacting with Pauli spins (Pauli field in short) [100, 101]:

$$\mathbf{J} = \sigma_{xy}^{\text{mixed}} \hat{\mathbf{z}} \times \nabla(\gamma H^z) \quad , \quad \mathbf{J}^e = e\mathbf{J} . \quad (8.9)$$

Here  $\gamma$  is gyromagnetic ratio. The related effect, which is determined by the same quantized parameter  $\sigma_{xy}^{\text{mixed}}$ , is the quantized response of the spin current, say the current  $\mathbf{J}^z$  of the  $z$  component of spin, to the gradient of chemical potential [102]. In the electrically charged systems this corresponds to the quantized response of the spin current to transverse electric field:

$$\mathbf{J}^z = \sigma_{xy}^{\text{mixed}} \hat{\mathbf{z}} \times \nabla\mu = e\sigma_{xy}^{\text{mixed}} \hat{\mathbf{z}} \times \mathbf{E} . \quad (8.10)$$

This kind of mixed Hall effect is now used in spintronics [157].

Finally there is a pure spin Hall effect – the quantized response of the spin current to transverse gradient of magnetic field [100, 101, 158, 159]:

$$\mathbf{J}^z = \sigma_{xy}^{\text{spin/spin}} \hat{\mathbf{z}} \times \nabla(\gamma H^z) . \quad (8.11)$$

All these parameters  $\sigma_{xy}$  are quantized being expressed via topological charges  $\tilde{N}_3$  and  $\tilde{N}_3^K$ .

## B. 3+1 fully gapped states: $^3\text{He-B}$ and quantum vacuum

In the asymmetric phase of Standard Model, there is no mass protection by topology and all the fermions become massive, i.e. Standard Model vacuum becomes the fully gapped insulator. In quantum liquids, the fully gapped three-dimensional system with time reversal symmetry and nontrivial topology is represented by another phase of superfluid  $^3\text{He}$  – the  $^3\text{He-B}$ . Its topology is also supported by symmetry and gives rise to the 2D gapless quasi-particles living at interfaces between vacua with different values of the topological invariant or on the surface of  $^3\text{He-B}$  [97, 98, 120, 144]. It is important that  $^3\text{He-B}$  belongs to the same topological class as the vacuum of Standard Model in its present insulating phase [99]. The topological classes of the  $^3\text{He-B}$  states can be represented by the following simplified Green's function:

$$G^{-1}(\omega, \mathbf{p}) = ip_0 + \tau_3 \left( \frac{p^2}{2m} - \mu \right) + \tau_1 (\sigma_x c_x p_x + \sigma_y c_y p_y + \sigma_z c_z p_z) . \quad (8.12)$$

In the isotropic  $^3\text{He-B}$  all ‘speeds of light’ are equal,  $|c_x| = |c_y| = |c_z| = c$ . The vacuum of free Dirac particles is obtained in the limit  $1/m = 0$ .

In the fully gapped systems, the Green's function has no singularities in the whole 4-dimensional space  $(p_0, \mathbf{p})$ . That is why we are able to use the Green's function at  $p_0 = 0$ . The topological invariant relevant for  $^3\text{He-B}$  and for quantum vacuum with massive Dirac fermions is:

$$N^K = \frac{e_{ijk}}{24\pi^2} \text{tr} \left[ \int_{\omega=0} d^3p \, K \, G \partial_{p_i} G^{-1} G \partial_{p_j} G^{-1} G \partial_{p_k} G^{-1} \right] . \quad (8.13)$$

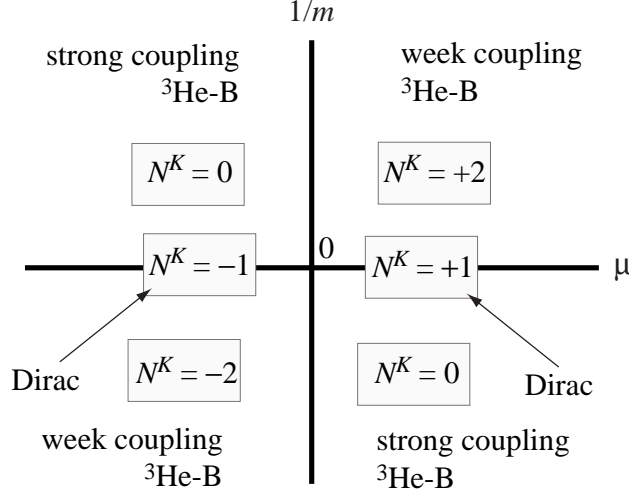


FIG. 12: Phase diagram of topological states of  ${}^3\text{He-B}$  in equation (8.12) in the plane  $(\mu, 1/m)$  for the speeds of light  $c_x > 0$ ,  $c_y > 0$  and  $c_z > 0$ . States on the line  $1/m = 0$  correspond to the Dirac vacua, which Hamiltonian is non-compact. Topological charge of the Dirac fermions is intermediate between charges of compact  ${}^3\text{He-B}$  states. The line  $1/m = 0$  separates the states with different asymptotic behavior of the Green's function at infinity:  $G^{-1}(\omega = 0, \mathbf{p}) \rightarrow \pm \tau_3 p^2 / 2m$ . The line  $\mu = 0$  marks topological quantum phase transition, which occurs between the weak coupling  ${}^3\text{He-B}$  (with  $\mu > 0$ ,  $m > 0$  and topological charge  $N^K = 2$ ) and the strong coupling  ${}^3\text{He-B}$  (with  $\mu < 0$ ,  $m > 0$  and  $N^K = 0$ ). This transition is topologically equivalent to quantum phase transition between Dirac vacua with opposite mass parameter  $M = \pm|\mu|$ , which occurs when  $\mu$  crosses zero along the line  $1/m = 0$ . The interface which separates two states contains single Majorana fermion in case of  ${}^3\text{He-B}$ , and single chiral fermion in case of relativistic quantum fields. Difference in the nature of the fermions is that in Bogoliubov-de Gennes system the components of spinor are related by complex conjugation. This reduces the number of degrees of freedom compared to Dirac case.

with matrix  $K = \tau_2$  which anti-commutes with the Green's function at  $p_0 = 0$ . In  ${}^3\text{He-B}$ , the  $\tau_2$  symmetry is combination of time reversal and particle-hole symmetries; for Standard Model the matrix  $\tau_2 = \gamma_5 \gamma^0$ . Note that at  $p_0 = 0$  the symmetry of the Green's function is enhanced, and thus there are more matrices  $K$ , which commute or anti-commute with the Green's function, than at  $p_0 \neq 0$ .

Fig. 12 shows the phase diagram of topological states of  ${}^3\text{He-B}$  in the plane  $(\mu, 1/m)$ . On the line  $1/m = 0$  one obtains the free Dirac fermions ( $n_L = n_R = 1$ ) with the mass parameter  $M = -\mu$ . The conventional Dirac vacuum of free fermions has topological charge

$$N^K = \text{sign}(M). \quad (8.14)$$

The real superfluid  ${}^3\text{He-B}$  lives in the weak-coupling corner of the phase diagram:  $\mu > 0$ ,  $m > 0$ ,  $\mu \gg mc^2$ . However, in the ultracold Fermi gases with triplet pairing the strong coupling limit is possible near the Feshbach resonance [160]. When  $\mu$  crosses zero the topological quantum phase transition occurs, at which the topological charge  $N^K$  changes from  $N^K = 2$  to  $N^K = 0$ .

There is an important difference between  ${}^3\text{He-B}$  and Dirac vacuum. The space of the Green's function of free Dirac fermions is non-compact:  $G$  has different asymptotes at  $|\mathbf{p}| \rightarrow \infty$  for different directions of momentum  $\mathbf{p}$ . As a result, the topological charge of the

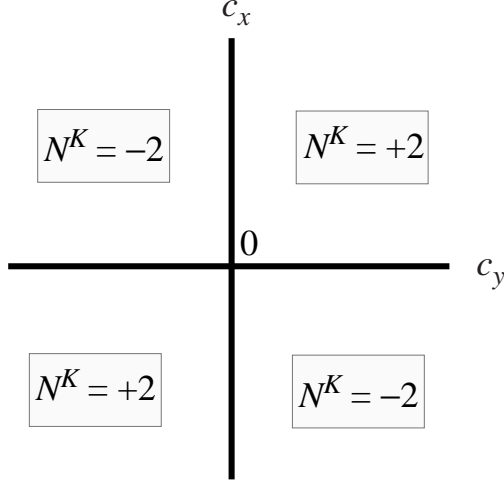


FIG. 13: Phase diagram of  ${}^3\text{He-B}$  states at fixed  $c_z > 0$ ,  $\mu > 0$  and  $m > 0$ . At the phase boundaries the vacuum is gapless and corresponds to the 3+1 planar phase. The interface between the gapped states with different winding number  $N^K$  contains Majorana fermions. In cosmology such interface would correspond to the vierbein wall, where the metric is degenerate [8].

interacting Dirac fermions depends on the regularization at large momentum.  ${}^3\text{He-B}$  can serve as regularization of the Dirac vacuum, which can be made in the Lorentz invariant way [99]. One can see from Fig. 12, that the topological charge of free Dirac vacuum has intermediate value between the charges of the  ${}^3\text{He-B}$  vacua with compact Green's function. On the marginal behavior of free Dirac fermions see Refs. [8, 144, 161, 162].

The vertical axis in Fig. 12 separates the states with the same asymptote of the Green's function at infinity. The abrupt change of the topological charge across the line,  $\Delta N^K = 2$ , with fixed asymptote shows that one cannot cross the transition line adiabatically. This means that all the intermediate states on the line of this QPT are necessarily gapless. For the intermediate state between the free Dirac vacua with opposite mass parameter  $M$  this is well known. But this is applicable to the general case with or without relativistic invariance: the gaplessness is protected by the difference of topological invariants on two sides of transition.

Fig. 13 shows phase diagram of topological states of  ${}^3\text{He-B}$  in the plane  $(c_x, c_y)$  at fixed  $c_z > 0$ ,  $\mu > 0$  and  $m > 0$ . When one of the components of speed of light is nullified the state becomes gapless. If, say,  $c_x$  approaches zero, two pairs of point nodes appear in this intermediate state at points  $\mathbf{p} = \pm(0, 0, p_F)$ , where  $p_F = (2\mu m)^{1/2}$ . The number of point nodes is related to the difference in topological invariant  $N^K$  of the states on two sides of the transition:

$$N_{\text{point nodes}} = |N^K(c_x > 0) - N^K(c_x < 0)| = 4. \quad (8.15)$$

The intermediate state is the 3+1 planar phase in (7.12), and the corresponding gapless fermions in this state are characterized by the topological invariant  $N_3^K$  protected by discrete symmetry.

The gaplessness of the intermediate state on phase diagram leads to the other related phenomenon. The two-dimensional interface (brane), which separates two domains with different  $N^K$ , contains fermion zero modes, 2+1 massless fermions. The number of zero



modes is determined by the difference  $\Delta N^K$  between the topological charges of the vacua on two sides of the interface [98, 144]. This is similar to the index theorem for the number of fermion zero modes on cosmic strings, which is determined by the topological winding number of the string [145].

Superfluid  $^3\text{He-B}$  demonstrates that the interface between the bulk states with the same topological charge may also contain fermion zero modes [97, 98]. For example, the state with positive  $c_x, c_y$  and the state with negative  $c_x, c_y$  in Fig. 13 have the same topological charge  $N^K = +2$ . But the interface between these bulk states is gapless. Moreover, the density of states of these irregular fermion zero modes is larger than the density of states of regular fermion zero modes living on the interface between the bulk states with different  $N^K$ . This is not very surprising, since the irregular gapless states of co-dimension 1 may appear in many systems. Nodes of co-dimension 1 are the most stable topological objects, and they may emerge even in bulk superconductors [163, 164]. They also may exist on the surface of non-topological insulators and superconductors, and at the interface with  $\Delta N^K = 0$  as it happens in  $^3\text{He-B}$ .

The  $^3\text{He-B}$  topology can be also extended to the lattice models explored in QCD, where the four dimensional Brillouin zone is used. The lattice model with fully gapped Wilson fermions is described both by  $\tilde{N}_3$  and  $\tilde{N}_5$  topological invariants, which give rise to two index theorems for gapless fermions in the intermediate states [122, 165]. The topological invariant  $\tilde{N}_5$  has been also used for description of fully gapped vacua in 4+1 systems [8, 146–148]. They give rise to the 3+1 gapless fermion zero modes living at the 3+1 interfaces between the massive vacua. This provides another topological scenario for emergent chiral relativistic fermion, accompanied by relativistic quantum field theory and gravity.

## IX. DISCUSSION

There is a fundamental interplay of symmetry and topology in physics, both in condensed matter and relativistic quantum fields. Traditionally the first role was played by symmetry (symmetry classification of crystals, liquid crystals, magnets, superconductors, superfluids, etc.). The phenomenon of spontaneously broken symmetry remains one of the major tools in physics. In particle physics the chain of successive phase transitions is suggested in which the large symmetry group existing at high energy is reduced at low energy. This symmetry principle is in the basis of the Grand Unification Theory (GUT). It also gives rise to the classification of the topological defects which arise due to spontaneous symmetry breaking. In this approach, symmetry is primary, while topology is secondary being fully determined by the broken symmetry.

The last decades demonstrated the opposite tendency in which topology is primary (the reviews can be found in Refs. [8, 83, 101]). The unconventional properties of superfluid  $^3\text{He-A}$ , which were found in early seventies, demonstrated that the topology in momentum space is the main characteristics of the ground states of the system at zero temperature ( $T = 0$ ), in other words it is the characteristics of quantum vacua. It demonstrated that the gaplessness of fermions is protected by topology, and thus is not sensitive to the details of the microscopic (trans-Planckian) physics. Irrespective of the deformation of the parameters of the microscopic theory, the value of the gap (mass) in the energy spectrum of these fermions remains strictly zero. This solves the main hierarchy problem: for these classes of fermionic vacua the masses of elementary particles are naturally small.

The vacua, which have nontrivial topology in momentum space, are now called the topo-

logical matter (topological superfluids, superconductors, insulators, semi-metals, etc.). The momentum-space topological invariants determine universality classes of the topological matter and the type of the effective theory which emerges at low energy and low temperature. In many cases they also give rise to emergent symmetry. Examples are provided by the nodes of the energy spectrum in momentum space: if they are protected by topology they give rise to emergent symmetries such as Lorentz invariance, and to emergent phenomena such as gauge and gravitational fields. Contrary to the GUT scheme, in the anti-GUT scheme the symmetry is secondary, being emergent in the low-energy corner due to topology. If this is true, then it is topology of the quantum vacuum, which gives rise to the fermionic matter in our Universe.

However, this is not the whole story. It appears that in many systems (including condensed matter and relativistic quantum vacua), the topological invariants are trivial, i.e. they have zero values. Nevertheless these systems remain to be topological: the underlying discrete symmetry of the vacuum may transform the trivial topological invariant into the nontrivial one. This is the case when both topology and symmetry are equally important. This implies that if the Standard Model and gravity are effective theories, the underlying physics must contain discrete symmetries. Their role is extremely important. The main role is to prohibit the cancellation of the Fermi points with opposite topological charges in the Standard Model vacuum. As a side effect, in the low-energy corner discrete symmetries are transformed into gauge symmetries and give rise to effective non-Abelian gauge fields. In particular, the underlying  $Z_2$  symmetry produces the effective  $SU(2)$  gauge field [8]. Discrete symmetries also reduce the number of the massless gauge bosons and the number of effective metric fields. To justify the Fermi point scenario of emergent physical laws, one should find such discrete symmetry of the microscopic vacuum which leads in the low energy corner to one of the GUT or Pati-Salam models.

The discovery of the quantum Hall effect (QHE) in 1980 [156] triggered the further consideration of condensed matter systems using topological methods [161, 166–169]. The topological invariant for QHE effect in terms of the Green's function was introduced in relativistic  $2+1$  quantum electrodynamics [115–117]. This invariant is responsible not only for ordinary QHE but also for the intrinsic QHE, when the Hall conductance is quantized even in the absence of external magnetic field [118].

Such intrinsic QHE can be realized for example in thin films of superfluid  $^3\text{He}$ . These quasi-two-dimensional systems serve as an example of the fully gapped  $2+1$  systems, whose integer valued topological invariant is in the origin of quantization of physical parameters [100, 112, 150]. There are two phases which may exist in a thin film of liquid  $^3\text{He}$ , both phases are fully gapped in film. These are chiral superfluid  $^3\text{He-A}$  and the planar phase with time reversal symmetry. This symmetry supports the topological invariant which is responsible for intrinsic quantum Hall and spin-Hall effects [100, 101], thus providing us with the important example of the interplay of topology and symmetry in topological medium.

Recent observation of topological insulators [170] gave new impulse for study of the topological phases in  $3+1$  systems, such as superfluid  $^3\text{He-B}$  which is the condensed matter counterpart of Standard Model vacuum in its massive phase. Superfluid  $^3\text{He-B}$  has Majorana bound states on the surface of the liquid. Andreev bound states on the surface or interface of  $^3\text{He-B}$  were discussed theoretically or probed experimentally [97, 171–177]. However, the Majorana signature of these states has not yet been reported experimentally. One of the possible tools is NMR, which requires an external magnetic field. The effect of magnetic field on Majorana fermions has been recently discussed in [120, 177]. Majorana fermions

become massive, with mass being proportional to magnetic field, which violates the time reversal symmetry.

Vacuum of Standard Model is a topological 3+1 medium. Both known states of the quantum vacuum of Standard Model have non-trivial topology. The insulating state is described by nonzero value of topological invariant  $\tilde{N}_3^K$  and the topology of this state is similar to that of superfluid  $^3\text{He-B}$ . The semi-metal state is described by nonzero value of topological invariant  $N_3^K$  and the topology of this state is similar to that of superfluid planar phase in 3+1 dimensions. Both invariants are supported by symmetry. Superfluid phases of  $^3\text{He}$  ( $^3\text{He-B}$ ,  $^3\text{He-A}$  and planar phase) thus provide a close connection with relativistic quantum fields.

These phases also give generic examples of different classes of topological media in dimensions  $2 + 1$  and  $3 + 1$  which experience topological stability of gap nodes; topological edge states on the surface of fully gapped insulators; Majorana fermions; topological quantum phase transitions; intrinsic quantum Hall and spin-Hall effects; quantization of physical parameters; chiral anomaly; topological Chern-Simons and Wess-Zumino actions; etc.

These examples of topological medium demonstrate the important role of both topology and symmetry. That is why we need the general classification of topological matter in terms of its symmetry and topology. The attempt of such classification was made in [143, 178]. However, only non-interacting systems were considered, and also not all the symmetries were exploited, including the approximate symmetries. Examples of latter are again provided by the superfluid phases of  $^3\text{He}$ , where the symmetry is enhanced due to relative smallness of the spin-orbit interactions. One task should be to consider symmetry classes, including crystal symmetry classes, magnetic classes, superconductivity classes, etc., and to find out what are the topological classes of Green's function which are allowed within a given symmetry class. Then one should find out what happens when the symmetry is smoothly violated, etc. The Green's function matrices with spin and band indices must be used for this classification, since they take into account interaction. In this way we may finally obtain the full classification of topological matter, including insulators, superconductors, magnets, liquids, and vacua of relativistic quantum fields.

Finally this experience must be used for the investigation of the topologically nontrivial quantum vacua in relativistic quantum field theories, where the topology in momentum space is becoming the important tool (see e.g. [8, 81, 115, 116, 122, 140, 146–148, 165, 179]).

On the other hand, the gravitational properties of quantum vacuum do not depend much on the topology of the vacuum. In any self-sustained vacuum, gapless or gapped, topological or trivial, the gravitating energy of the vacuum is strictly zero if the vacuum is perfect and is isolated from environment. This is the consequence of thermodynamics, which is not sensitive to the structure of the quantum vacuum. Nullification of  $\rho_{\text{vac}}$  occurs in any equilibrium system at  $T = 0$ , relativistic or non-relativistic. This solves the main cosmological constant problem: if the quantum vacuum belongs to self-sustained media,  $\Lambda$  is naturally small. According to this view, cosmology is the process of relaxation of the Universe to its equilibrium vacuum state, and this process does depend on the momentum space topology of the quantum vacuum.

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